On the graph of multiplication lattice modules

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Abstract. Let $L$ be a $C$-lattice and $M$ be a lattice module over $L$. In this paper, we introduce the graph $G(M)$ for $M$ and investigate the relationship between the algebraic properties of $M$ and the properties of its associated graph when $M$ is multiplication.

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1. Introduction

The graphs have played important roles in the study of commutative rings, modules over a commutative ring and their applications in coding theory, automata theory etc. There are several mathematicians who studied the interrelations between the algebraic properties and the graph-theoretic properties for various algebraic structures viz. group, ring, module etc.

B. Csakany and G. Pollak [13], studied the graph of subgroups of a finite group as an extension of study of J. Bosak [5]. I. Beck [6] introduced the idea of a zero-divisor graph of a commutative ring $R$ with unity. Inspired by his work, lot of study on the graph of ideals has been carried out (see [2], [7]-[8], [14]). Recently, H. Ansari-Toroghy and S. Habibi generalized the annihilating-ideal graph of $R$ to submodules of $M$ and defined the graph $\overline{AG}(M)$, called the annihilating-submodule graph (see [4]).

In this paper, we introduce graph $G(M)$ of multiplication lattice module $M$ with vertex set $V(G(M)) = \{N \in M | N$ is non-small element\}. In this graph, two distinct vertices $N, K$ are adjacent if and only if $N \land K$ is a non-small element of $M$. Also, study some algebraic properties of a lattice module $M$ by using graph $G(M)$.

By a $C$-lattice, we mean a multiplicative lattice $L$, with least element $0_L$ and greatest element $1_L$ which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset $C$ of compact elements of $L$. A proper element $p$ of $L$ is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$. A proper element $p$ of $L$ is said to be quasi-prime if $a \land b \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Joshi and Sarode [20] introduced the multiplicative zero-divisor graph of a multiplicative lattice and studied Beck-like coloring of such graphs and proved that for such graphs, the chromatic number and the clique number need not be equal. On the other hand, if a multiplicative lattice $L$ is reduced, then the chromatic number and the clique number of the multiplicative zero-divisor graph of $L$ are equal, which extends the result of Behboodi and Rakeei [8] and Aalipour et al. [1].

A complete lattice $M$ with smallest element $0_M$ and greatest element $1_M$ is said to be a lattice module over the multiplicative lattice $L$ or $L$-module if there is a multiplication between elements of $M$ and $L$, denoted by $aN$ for $a \in L$ and $N \in M$, which satisfies the following properties:

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(i) \((ab)N = a(bN)\);

(ii) \((\vee \alpha a_\alpha)(\vee \beta N_\beta) = \vee \alpha, \beta (a_\alpha N_\beta)\);

(iii) \(1_L N = N\);

(iv) \(0_L N = 0_M\); for \(a, b, a_\alpha \in L\) and for \(N, N_\beta \in M\).

For \(N \in M\) and \(b \in L\), denote \((N : b) = \vee \{K \in M | bK \leq N\}\). If \(A, B \in M\) then \((A : B) = \vee \{x \in L | BX \leq A\}\).

An element \(N\) of \(M\) is said to be compact if \(N \leq \bigvee_{\alpha \in I} A_\alpha (I\) is an indexed set\) implies \(N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}\) for some subset \(\{\alpha_1, \alpha_2, \cdots, \alpha_n\}\) of \(I\).

F. Calliap et al. ([15], [16]) investigated properties of multiplication lattice modules. A lattice module \(M\) over a multiplicative lattice \(L\) is said to be multiplication lattice module, if for every element \(N \in M\) there exists an element \(a \in L\) such that \(N = a_1M\). A lattice module \(M\) is a multiplication lattice module if and only if \(N = (N : 1_M)1_M\) for each \(N \in M\) (see [15]). A proper element \(K \in M\) is said to be a small if for every element \(N\) of \(M\) such that \(K \vee N = 1_M\) implies \(N = 1_M\), otherwise \(N\) is said to be a non-small element of \(M\) (see [16]).

Eaman A. Al-Khouja [3] studied the relationship between the maximal (prime) elements of \(M\) and the maximal (prime) elements of \(L\). A proper element \(N\) of \(M\) is said to be maximal if for every element \(B \in M\) with \(N \leq B\), implies either \(N = B\) or \(B = 1_M\). We denote the set of all maximal elements of \(M\) by \(\text{Max}(M)\).

A proper element \(N\) of \(M\) is said to be prime if \(aX \leq N\) implies \(X \leq N\) or \(a_1M \leq N\), i.e., \(a \leq (N : 1_M)\) for every \(a \in L\) and \(X \in M\). Note that, every maximal element of \(M\) is prime and also, if \(N\) is a prime element of \(M\) then \((N : 1_M)\) is a prime element of \(L\) (see [3]).

Further for more information and all these concepts on multiplicative lattices, lattice modules and graph theory the reader may refer ([9]-[12], [18], [22]).

\section{2. Graph \(G(M)\) of a Lattice Module \(M\)}

**Definition 2.1.** Let \(M\) be a lattice module over a \(C\)-lattice \(L\). We define the graph \(G(M)\) of \(M\) with vertex set \(V(G(M)) = \{N \in M | N\) is non-small element\} such that two distinct vertices \(N, K\) are adjacent if and only if \(N \wedge K\) is a non-small element of \(M\).

**Example 2.2.**

![Lattice Module M](image-url)
The following result is useful throughout the paper.

**Lemma 2.3.** [19] Let $M$ be a lattice module over a multiplicative lattice $L$. Then for $a \in L$ and $A, B, C \in M$, following holds:

(i) If $A \leq B$ then $(A : a) \leq (B : a)$.

(ii) If $A \leq B$ then $(A : C) \leq (B : C)$.

(iii) $(B \land C : A) = (B : A) \land (C : A)$.

(iv) $a \leq (aA : A)$.

(v) $a(A : a) \leq A$.

(vi) $(A : B)B \leq A$.

**Remark 2.4.** For $N, K \in M$. If $N \leq K$ and $N$ is non-small then by Lemma 2.3(2), $K$ is also a non-small element of $M$.

Following Lemma follows from Remark 2.4.

**Lemma 2.5.** For $N, K \in V(G(M))$. If $N \leq K$ then $\text{deg}(N) \leq \text{deg}(K)$.

**Lemma 2.6.** [21] Let $M$ be a lattice module over a $C$-lattice $L$ with $1_M$ is compact. Then for every proper element $X \in M$, there exists a maximal element $N$ of $M$ such that $X \leq N$.

Let $M$ be a lattice module over a multiplicative lattice $L$. The Jacobson radical $J(M) = \bigwedge \text{Max}(M)$ (see [3]).

**Theorem 2.7.** Let $M$ be a lattice module over a $C$-lattice $L$ with $1_M$ is compact and $N \in M$. Then $N$ is small if and only if $N \leq J(M)$.

**Proof.** Suppose that $N$ is a small element of $M$ and $N \notin J(M)$. Then there exists $K \in \text{Max}(M)$ such that $N \not\leq K$ which implies $N \lor K = 1_M$. Since $N$ is small, we have $K = 1_M$, a contradiction, and consequently, $N \leq J(M)$.

Conversely, suppose that $N \leq J(M)$ and $N$ is non-small element of $M$. Then there exists a proper element $K$ of $M$ such that $N \lor K = 1_M$. Since $M$ is a lattice module over a $C$-lattice $L$ with $1_M$ is compact, by Lemma 2.6, there exists $P \in \text{Max}(M)$ such that $K \leq P$ therefore $1_M = N \lor K \leq N \lor P$ hence $N \lor P = 1_M$. But $N \leq J(M)$; therefore, $N \leq P$ and hence, $P = 1_M$, a contradiction, consequently, $N$ is a small element of $M$.

Following definition is due to Eaman A. Al-Khoujja [3].

**Definition 2.8.** A lattice module $M$ over a multiplicative lattice $L$ is said to be a local if it has only one maximal element.
A graph whose vertex set is empty is a *null graph* and a graph whose edge set is empty is an *empty graph* (see [18]).

Following Corollary follows from Theorem 2.7.

**Corollary 2.9.** Let $M$ be a lattice module over a C-lattice $L$ with $1_M$ is compact. Then $G(M)$ is a null graph if and only if $M$ is local.

In the rest of paper we assume that $G(M)$ is a non-null graph.

**Lemma 2.10.** Let $M$ be a multiplication lattice module over a C-lattice $L$ and $N, K \in M$. If $P$ is a prime element of $M$ with $N \cap K \leq P$ then either $N \leq P$ or $K \leq P$.

**Proof.** Suppose that $P$ is a prime element of $M$ with $N \cap K \leq P$. Then by Lemma 2.3(2) and (3),

$$ (N : 1_M) \cap (K : 1_M) = (N \cap K : 1_M) \leq (P : 1_M). $$

Since $P$ is prime, $(P : 1_M)$ is prime and hence quasi-prime element of $L$ therefore $(N : 1_M) \cap (K : 1_M) \leq (P : 1_M)$ implies either $(N : 1_M) \leq (P : 1_M)$ or $(K : 1_M) \leq (P : 1_M)$ and so either $(N : 1_M)1_M \leq (P : 1_M)1_M$ or $(K : 1_M)1_M \leq (P : 1_M)1_M$. Hence $N \leq P$ or $K \leq P$ because $M$ is a multiplication lattice module.

**Corollary 2.11.** Let $M$ be a multiplication lattice module over a C-lattice $L$ with $\text{Max}(M) = \{K_{i \in I}\}$ $(I$ is an indexed set). Then for all non-empty proper finite subset $\Lambda$ of $I$, $\wedge_{i \in \Lambda} K_i$ is non-small.

**Proof.** Suppose that for a non-empty proper finite subset $\Lambda$ of $I$, $\wedge_{i \in \Lambda} K_i$ is a small element of $M$. Then by Theorem 2.7, $\wedge_{i \in \Lambda} K_i \leq J(M) \leq K_i$ where $j \in I$ and $j \notin \Lambda$. Since every maximal is a prime element of $M$, by Lemma 2.10, for some $i \in \Lambda$, $K_i \leq K_j$, a contradiction, consequently, $\wedge_{i \in \Lambda} K_i$ is a non-small element of $M$.

**Theorem 2.12.** Let $M$ be a lattice module over a C-lattice $L$ with $1_M$ is compact. Then following statements are equivalent.

(i) $G(M)$ is not connected.

(ii) $|\text{Max}(M)| = 2$.

(iii) $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs.

**Proof.** 1) $\Rightarrow$ 2) Suppose that $|\text{Max}(M)| > 2$. Since $G(M)$ is not connected, we have components $G_1$ and $G_2$ of $G(M)$. Let $N, K \in M$ such that $N \in V(G_1)$ and $K \in V(G_2)$. It is clear that $N \cap K$ is small. Since $|\text{Max}(M)| > 2$, by Lemma 2.6, there exist $S_1, S_2 \in \text{Max}(M)$ such that $N \leq S_1$ and $K \leq S_2$. Note that $N \cap S_1$ and $K \cap S_2$ are non-small because $(N \cap S_1) \lor S_2 = 1_M$ and $(K \cap S_2) \lor S_1 = 1_M$.

Case 1: If $S_1 = S_2$ then

$$\begin{array}{ccc}
N & s_1 & s_2 & K \\
\hline
& s_1 & s_2 & K
\end{array}$$

is a path between $G_1$ and $G_2$, a contradiction.

Case 2: If $S_1 \neq S_2$. Since $|\text{Max}(M)| > 2$ then by Lemma 2.11, $S_1 \cap S_2$ is non-small. Therefore

$$\begin{array}{ccc}
N & s_1 & s_2 & K \\
\hline
& s_1 & s_2 & K
\end{array}$$

is a path, a contradiction, consequently, $|\text{Max}(M)| = 2$.

2) $\Rightarrow$ 3) Suppose that $\text{Max}(M) = \{S_1, S_2\}$ and $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$, $G_2(M)$ are subgraphs of $G(M)$. Since $M$ is a lattice module over a C-lattice $L$ with $1_M$ is compact and $\text{Max}(M) = \{S_1, S_2\}$, by lemma 2.6, $G_1 = \{N \in M|N \leq S_1$ and $N$ is non-small$\}$ and $G_2 = \{X \in M|X \leq S_2$ and
Similarly, for any $N, K \in V(G(M))$, $N, K$ are adjacent, i.e., $N \land K$ is non-small. Suppose that $N \land K$ is small. Then by Theorem 2.7, $N \land K \leq S_1 \land S_2 = J(M)$ therefore by Lemma 2.10, either $N \leq S_2$ or $K \leq S_2$ because every maximal is a prime element of $M$ and $J(M) \leq S_2$. This implies either $N$ or $K$ is small, a contradiction to $N, K \in V(G_1(M))$, consequently, $N$ and $K$ are adjacent, i.e., $N \land K$ is non-small. Similarly, for any $X, Y \in V(G_2(M))$, $X \land Y$ is non-small. Hence $G_1(M)$ and $G_2(M)$ are complete subgraph.

Now to prove that $G_1(M)$ and $G_2(M)$ are disjoint, consider $N \in V(G_1(M))$ and $X \in V(G_2(M))$. If $N$ and $X$ are adjacent, then $N \land X$ is non-small. But $N \leq S_1$ and $X \leq S_2$ therefore $N \land X \leq S_1 \land S_2 = J(M)$ and so by Theorem 2.7, $N \land X$ is small, a contradiction. Consequently, $G_1(M)$ and $G_2(M)$ are disjoint subgraphs.

3) $\Rightarrow$ 1) Obvious.

A clique of a graph is its maximal complete subgraph and the clique number of $G$ is the number of vertices in the largest clique of a graph $G$, denoted by $\omega(G)$.

Following Corollary follows from Corollary 2.11 and Theorem 2.12.

**Corollary 2.13.** Let $M$ be a multiplication lattice module over a C-lattice $L$ with $1_M$ is compact and $G(M)$ is a non-empty graph. Then $\omega(G(M)) \geq |\text{Max}(M)|$.

For a graph $G$, $d(a, b)$ is the length of the shortest path connecting $a$ and $b$ and if there is no path between $a$ and $b$ then $d(a, b) = \infty$. The diameter of a graph $G$ is $\text{diam}(G) = \sup\{d(a, b)|a, b \in V(G)\}$.

**Theorem 2.14.** Let $M$ be a multiplication lattice module over a C-lattice $L$ with $1_M$ is compact. If $G(M)$ is a connected graph then $\text{diam}(G(M)) \leq 2$.

**Proof.** Suppose that $G(M)$ is a connected graph and $N, K \in V(G(M))$. If $N, K$ is adjacent then $d(N, K) = 1$. Suppose that $N, K$ are not adjacent vertices. Since $M$ is a lattice module over a C-lattice $L$ with $1_M$ is compact, by Lemma 2.6, there exist $S_1, S_2 \in \text{Max}(M)$ such that $N \leq S_1$ and $K \leq S_2$. If $N \land S_2$ is non-small then

$$
\begin{array}{c}
& N & \leq S_2 & K \\
\end{array}
$$

is a path and so $d(N, K) = 2$. Similarly, if $K \land S_1$ is non-small then $d(N, K) = 2$. Now, suppose that both $N \land S_2$ and $K \land S_1$ are small. Then by Theorem 2.7, $N \land S_2 \leq J(M)$ and $K \land S_1 \leq J(M)$. Since $G(M)$ is a connected graph, by Theorem 2.12, $|\text{Max}(M)| > 2$ therefore there exists $S_3 \in \text{Max}(M)$ other than $S_1, S_2$ such that $N \land S_2 \leq J(M) \leq S_3$ and $N \land S_1 \leq J(M) \leq S_3$. Note that $S_1, S_2 \notin S_3$, therefore by Lemma 2.10, $N \leq S_3$ and $K \leq S_3$ and hence there is a path

$$
\begin{array}{c}
& N & \leq S_3 & K \\
\end{array}
$$

between $N$ and $K$ of length 2, consequently, $\text{diam}(G(M)) \leq 2$.

The girth of a graph $G$ is the length of the shortest cycle in $G$ and it is denoted by $g(G)$ (see [18]).

**Theorem 2.15.** Let $M$ be a multiplication lattice module over a C-lattice $L$ with $1_M$ is compact. If $G(M)$ contains a cycle then $g(G(M)) = 3$.

**Proof.** Suppose that $G(M)$ contains a cycle. Since $G(M)$ is not a null graph, we have $|\text{Max}(M)| \geq 2$. Case-1: If $|\text{Max}(M)| = 2$, then by Theorem 2.12, $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs hence $g(G(M)) = 3$ because $G(M)$ contains a cycle.

Case-2: If $|\text{Max}(M)| \geq 3$. Then by Corollary 2.11, for $N_1, N_2, N_3 \in \text{Max}(M)$,
is a cycle in $G(M)$, consequently, $g(G(M)) = 3$.

A vertex $a$ in a connected graph $G$ is a cut vertex if $G - \{a\}$ is disconnected.

**Theorem 2.16.** Let $M$ be a multiplication lattice module over a $C$–lattice $L$ with $1_M$ is compact. If $G(M)$ is a connected graph then $G(M)$ has no cut vertex.

**Proof.** Suppose that $G(M)$ is a connected graph and $K \in V(G(M))$ is a cut vertex of $G(M)$. By definition $G(M) - \{K\}$ is not connected. This implies that for some $N, L \in V(G(M))$, $K$ lies in each path between $N$ and $L$. Since $G(M)$ is a connected graph, by Theorem 2.14, we have a path

$$\begin{array}{c}
N & \cdots & K & \cdots & L
\end{array}$$

of maximum length. Observe that, $K$ is maximal because if $K$ is not a maximal element of $M$ then by Lemma 2.6, there exists $S \in \text{Max}(M)$ such that $K \leq S$ and so $K \wedge N \leq S \wedge N$. Since $K$ and $N$ are adjacent,i.e, $K \wedge N$ is non-small, by Remark 2.4, $S$ and $N$ are also adjacent vertices. Similarly, we can say that $S$ and $L$ are adjacent vertices and hence there is a path

$$\begin{array}{c}
N & \cdots & S & \cdots & L
\end{array}$$

which is contradiction to the fact $K$ lies in each path between $N$ and $L$, consequently $K$ is maximal. Also, note that there exists a maximal element $S_i$ other than $K$ with $N \not\leq S_i$, because if for each $S_i \in \text{Max}(M)$, $N \leq S_i$, then $N \wedge K \leq J(M)$, which implies by Theorem 2.7 that $N \wedge K$ is small, a contradiction. On the same line, we have $S_j \in \text{Max}(M)$ other than $K$ with $L \not\leq S_j$. Since $N \wedge L$ small, by Theorem 2.7 and Lemma 2.10, for distinct $S_i, S_j \in \text{Max}(M)$ other than $K$, if $N \not\leq S_i$ and $L \not\leq S_j$ then $N \leq S_j$ and $L \leq S_i$. Hence

$$\begin{array}{c}
N & \cdots & S_j & \cdots & S_i & \cdots & L
\end{array}$$

is a path in $\Gamma(M) - \{K\}$, a contradiction, and consequently, $G(M)$ has no cut vertex.

**Theorem 2.17.** Let $M$ be a multiplication lattice module over a $C$–lattice $L$ with $1_M$ is compact and $|\text{Max}(M)| < \infty$. Then there is no vertex in $V(G(M))$ which is adjacent to every other vertex.

**Proof.** Suppose that there exists a vertex $N \in V(G(M))$ which is adjacent to every other vertex. Since $1_M$ is compact, by Lemma 2.6, there exists $S_i \in \text{Max}(M)$ such that $N \leq S_i$. By Corollary 2.11, for $S_j \in \text{Max}(M)$, $K = \wedge_{j \neq i} S_j$ is non-small. Since $N$ is adjacent to every other vertex, $N \wedge K$ is non-small. But $N \wedge K \leq \wedge_{j \neq i} S_j \wedge S_i = J(M)$ therefore by Theorem 2.7, $N \wedge K$ is small, a contradiction, consequently, there is no vertex in $V(G(M))$ which is adjacent to every other vertex.

Following Corollary follows from Theorem 2.17.

**Corollary 2.18.** Let $M$ be a multiplication lattice module over a $C$–lattice $L$ with $1_M$ is compact and $|\text{Max}(M)| < \infty$. Then $G(M)$ is not a complete graph.

For $a \in V(G)$, the neighborhood of $a$ is the set of vertices which are adjacent to $a$. A vertex of a graph $G$ is said to be pendant if its neighborhood contains exactly one vertex.
Theorem 2.19. Let $M$ be a multiplication lattice module over a $C$-lattice $L$ with $1_M$ is compact. If $G(M)$ has a pendant vertex then $|\text{Max}(M)| = 2$.

Proof. Suppose that $N \in V(G(M))$ is a pendant vertex and $|\text{Max}(M)| \geq 3$. Then by Corollary 2.11, for distinct $S_1, S_2 \in \text{Max}(M)$, $S_1 \cup S_2$ is non-small. This implies that for each $S_i$, $\deg(S_i) \geq 2$ therefore $N \notin \text{Max}(M)$. Since $1_M$ is compact, by Lemma 2.6, there exists a maximal element $S_1$ of $M$ such that $N \leq S_1$ and hence $S_1$ is the only element that is adjacent to $N$ because $N$ is pendant. Thus for $S_2 \in \text{Max}(M)$, $N \cap S_2$ is small, therefore by Theorem 2.7, $N \cap S_2 \leq J(M)$ and hence for $S_1 \neq S_2$, $N \cap S_2 \leq S_1$. Since $S_1$ is prime and $S_2 \in \text{Max}(M)$, by Lemma 2.10, $N \leq S_2$, a contradiction, consequently, $|\text{Max}(M)| = 2$.

Corollary 2.20. Let $M$ be a multiplication lattice module over a $C$-lattice $L$ with $1_M$ is compact. If $G(M)$ has a pendant vertex then $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are disjoint and complete subgraphs.

Corollary 2.21. Let $M$ be a multiplication lattice module over a $C$-lattice $L$ with $1_M$ is compact. Then $G(M)$ is not a star graph.

Proof. Suppose that $G(M)$ is a star graph. Then by definition, $G(M)$ is connected and have a pendant vertex. Therefore by Theorem 2.12 and Theorem 2.19, $G(M)$ is not connected graph, a contradiction.

If the degree of each vertex is $r$ then graph $G$ is said to be $r$-regular graph.

Lemma 2.22. Let $M$ be a multiplication lattice module over a $C$-lattice $L$ with $1_M$ is compact. If $G(M)$ is a $r$-regular graph, then $|\text{Max}(M)| = 2$ and $|V(G(M))| = 2r + 2$.

Proof. Suppose that graph $G(M)$ is a $r$-regular and $|\text{Max}(M)| \geq 3$. By Lemma 2.5, $\deg(S_1 \cup S_2) \leq \deg(S_1)$ where $S_1, S_2 \in \text{Max}(M)$. Note that $\deg(S_1 \cap S_2) \neq \deg(S_1)$, because by Corollary 2.11, $S_1 \cap S_2$ is adjacent to $S_1$, however by Theorem 2.7, $\cap S_1 \neq S_2$ is not adjacent to $S_1 \cap S_2$. This implies that $\deg(S_1 \cup S_2) \leq \deg(S_1)$ which is a contradiction to the fact that $G(M)$ is a $r$-regular graph and therefore $|\text{Max}(M)| \leq 2$. Since $G(M)$ is not a null graph, we have $|\text{Max}(M)| \neq 1$ hence $|\text{Max}(M)| = 2$. Let $\text{Max}(M) = \{N_1, N_2\}$. Then by Theorem 2.12, $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs with $V(G_1) = \{S_i \in M | S_i \leq N_1\}$ and $V(G_2) = \{S_j | S_j \leq N_2\}$. Since $G(M)$ is a $r$-regular graph, $V(G_i) = r + 1$, consequently, $|V(G(M))| = 2r + 2$.

A subset $Z \subseteq V(G)$ is independent, if no two vertices in $Z$ are adjacent. The independence number $\alpha(G)$ is the maximum size of an independent set in $G$.

Theorem 2.23. Let $M$ be a multiplication lattice module over a $C$-lattice $L$ with $1_M$ is compact and $|\text{Max}(M)| < \infty$. Then $\alpha(G(M)) = |\text{Max}(M)|$.

Proof. Suppose that $\text{Max}(M) = \{S_1, S_2, \cdots, S_n\}$. Then for $i = 1, 2, \cdots, n$, $Y = \{\cap_{j=1, j \neq i}^n S_j\}$ is an independent set by Theorem 2.7. This implies that $\alpha(G(M)) \geq n$. Now to prove that $\alpha(G(M)) = n$, let $\alpha(G(M)) = m > n$ and let $X = \{N_1, N_2, \cdots, N_m\}$ be a maximal independent set. Note that for any $N_i \in X$, $N_i$ is a non-small element of $M$, therefore for some $S_i \in \text{Max}(M)$, $N_i \notin S_i$ by Theorem 2.7. Since $m > n$, by Pigeon hole principle, there exist $N_i, N_j \in X$ such that $N_i \notin S_i$ and $N_j \notin S_t$ where $1 \leq i, j \leq n$. But $X$ is an independent set and $N_i, N_j \in X$ therefore $N_i \cap N_j$ is small and so by Theorem 2.7, $N_i \cap N_j \leq J(M) \leq S_i$. Since $S_i$ is prime, by Lemma 2.10, either $N_i \leq S_i$ or $N_j \leq S_i$, a contradiction, consequently, $m = n = \alpha(G(M))$.

A $k$-partite graph is a graph whose vertices are or can be partitioned into $k$ different independent sets. A complete $k$-partite graph is a $k$-partite graph in which there is an edge between every pair of vertices from different independent sets (see [18]).
Theorem 2.24. Let $M$ be a multiplication lattice module over a $C$–lattice $L$ with $1_M$ is compact. Then $G(M)$ is not a complete $n$-partite graph.

Proof. Suppose that $G(M)$ is a complete $n$-partite graph with independent sets $X_1, X_2, \ldots, X_n$. Note that, every complete graph is necessarily connected, therefore by Theorem 2.12, $|\text{Max}(M)| \geq 3$, and so by Corollary 2.11, for any $N_i, N_j \in \text{Max}(M)$, $N_i \cap N_j$ is non-small. This implies that each $X_i$ contains at most one maximal element, therefore by Pigeon hole principle $|\text{Max}(M)| \leq n$. Note that $|\text{Max}(M)| = n$, indeed if $|\text{Max}(M)| = l < n$ then there exists set $X_l$ with no maximal element. By Corollary 2.11, for $N_j \in \text{Max}(M)$, $\wedge_{j \neq i} N_j$ is non-small. But $N_i \wedge (\wedge_{j \neq i} N_j) = J(M)$, therefore by Theorem 2.7, $\wedge_{j \neq i} N_j$ and $N_i$ are non-adjacent vertices and hence $\wedge_{j \neq i} N_j \in X_i$. Now, suppose that $N \in X_l$. Since $1_M$ is compact, By Lemma 2.6, there exists a maximal element $N_k$ such that $N \leq N_k$ and so $N$ is adjacent to $N_k$. Note that $G(M)$ is a complete $n$-partite graph and $N_k \in X_k$ therefore $N$ is adjacent to all vertices of $X_k$. Thus $N$ is adjacent to $\wedge_{j \neq k} N_j$, i.e., $N \wedge (\wedge_{j \neq k} N_j)$ is non-small, which is contradiction to the fact that $N \wedge (\wedge_{j \neq k} N_j) \leq N_k \wedge (\wedge_{j \neq k} N_j) = J(M)$, consequently, $|\text{Max}(M)| = n$. Now, let $H = \wedge_{j=3}^n N_j$, where $N_j \in \text{Max}(M)$. Then by Corrolary 2.11, $H \in V(G(M))$. Note that $H \wedge N_2 = \wedge_{j \neq 2} N_j$ is non-small, i.e., $H$ is adjacent to $N_1$. Similarly, $H$ is adjacent to $N_2$. Hence $H \notin X_1$ and $H \notin X_2$. Also, by Theorem 2.7, for each $i$ $(3 \leq i \leq n)$, $H \wedge N_i = H$ is non-small. This implies that $H$ is adjacent to all maximal element $N_i$ of $M$, hence for each $i$, $H \notin X_i$, a contradiction, consequently, $G(M)$ is not a complete $n$-partite graph.

References


REFERENCES


[18] Harary, F. Graph theory, Narosa, New Delhi.


