EURASIAN BULLETIN OF MATHEMATICS
EBM (2018), VOL. 1, NO. 3, 117-125

On the graph of multiplication lattice modules

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Abstract. Let L be a $C-$lattice and M be a lattice module over L. In this paper, we introduce the graph $G(M)$ for M and investigate the relationship between the algebraic properties of M and the properties of its associated graph when M is multiplication.
2010 Mathematics Subject Classifications: 05C38, 05C40, 06B75
Key Words and Phrases: Small element; Maximal element; Multiplication lattice module.

1. Introduction

The graphs have played important roles in the study of commutative rings, modules over a commutative ring and their applications in coding theory, automata theory etc. There are several mathematicians who studied the interrelations between the algebraic properties and the graph-theoretic properties for various algebraic structures viz. group, ring, module etc.
B. Csakany and G. Pollak [13], studied the graph of subgroups of a finite group as an extension of study of J. Bosak [5]. I. Beck [6] introduced the idea of a zero-divisor graph of a commutative ring R with unity. Inspired by his work, lot of study on the graph of ideals has been carried out (see [2], [7]-[8], [14]). Recently, H. Ansari-Toroghy and S. Habibi generalized the annihilating-ideal graph of R to submodules of M and defined the graph $\mathbb{AG}(M)$, called the annihilating-submodule graph (see [4]).
In this paper, we introduce graph $G(M)$ of multiplication lattice module M with vertex set $V(G(M)) = \{N \in M | N \text{ is non-small element}\}$. In this graph, two distinct vertices N, K are adjacent if and only if $N \wedge K$ is a non-small element of M. Also, study some algebraic properties of a lattice module M by using graph $G(M)$.

By a **C-lattice**, we mean a multiplicative lattice L, with least element 0_L and greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L. A proper element p of L is said to be *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$. A proper element p of L is said to be *quasi-prime* if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Joshi and Sarode [20] introduced the multiplicative zero-divisor graph of a multiplicative lattice and studied Beck-like coloring of such graphs and proved that for such graphs, the chromatic number and the clique number need not be equal. On the other hand, if a multiplicative lattice L is reduced, then the chromatic number and the clique number of the multiplicative zero-divisor graph of L are equal, which extends the result of Behboodi and Rakeei [8] and Aalipour et al. [1].

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or L-module if there is a multiplication between elements of M and L, denoted by aN for $a \in L$ and $N \in M$, which satisfies the following properties:

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(i) $(ab)N = a(bN)$;

(ii) $(\vee_\alpha a_\alpha)(\vee_\beta N_\beta) = \vee_{\alpha,\beta}(a_\alpha N_\beta)$;

(iii) $1_L N = N$;

(iv) $0_L N = 0_M$; for $a, b, a_\alpha \in L$ and for $N, N_\beta \in M$.

For $N \in M$ and $b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$. If $A, B \in M$ then $(A : B) = \vee\{x \in L | Bx \leq A\}$.

An element N of M is said to be *compact* if $N \leq \bigvee_{\alpha \in I} A_\alpha$($I$ is an indexed set) implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I.

F. Callialp et al. ([15], [16]) investigated properties of multiplication lattice modules. A lattice module M over a multiplicative lattice L is said to be *multiplication lattice module*, if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$. A lattice module M is a multiplication lattice module if and only if $N = (N : 1_M)1_M$ for each $N \in M$ (see [15]). A proper element $K \in M$ is said to be a *small* if for every element N of M such that $K \vee N = 1_M$ implies $N = 1_M$, otherwise N is said to be a *non-small* element of M (see [16]).

Eaman A. Al-Khouja [3] studied the relationship between the maximal (prime) elements of M and the maximal (prime) elements of L. A proper element N of M is said to be *maximal* if for every element $B \in M$ with $N \leq B$, implies either $N = B$ or $B = 1_M$. We denote the set of all maximal elements of M by $Max(M)$.

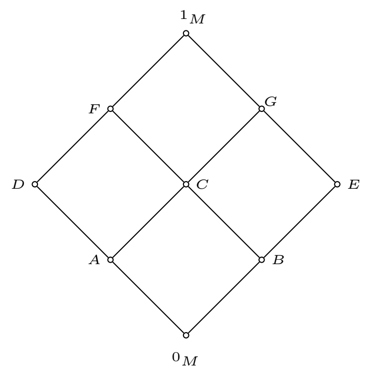
A proper element N of M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. Note that, every maximal element of M is prime and also, if N is a prime element of M then $(N : 1_M)$ is a prime element of L (see [3]).

Further for more information and all these concepts on multiplicative lattices, lattice modules and graph theory the reader may refer ([9]-[12], [18], [22]).

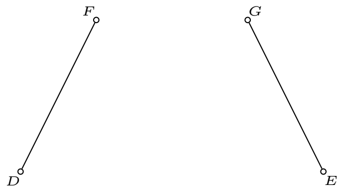
2. Graph $G(M)$ of a Lattice Module M

Definition 2.1. Let M be a lattice module over a C-lattice L. We define the graph $G(M)$ of M with vertex set $V(G(M)) = \{N \in M | N$ is non-small element$\}$ such that two distinct vertices N, K are adjacent if and only if $N \wedge K$ is a non-small element of M.

Example 2.2.



Lattice Module M



Graph $G(M)$

The following result is useful throughout the paper.

Lemma 2.3. *[19] Let M be a lattice module over a multiplicative lattice L. Then for $a \in L$ and $A, B, C \in M$, following holds:*

(i) *If $A \leq B$ then $(A : a) \leq (B : a)$.*

(ii) *If $A \leq B$ then $(A : C) \leq (B : C)$.*

(iii) *$(B \wedge C : A) = (B : A) \wedge (C : A)$.*

(iv) *$a \leq (aA : A)$.*

(v) *$a(A : a) \leq A$.*

(vi) *$(A : B)B \leq A$.*

Remark 2.4. For $N, K \in M$. If $N \leq K$ and N is non-small then by Lemma 2.3(2), K is also a non-small element of M.

Following Lemma follows from Remark 2.4.

Lemma 2.5. *For $N, K \in V(G(M))$. If $N \leq K$ then $deg(N) \leq deg(K)$.*

Lemma 2.6. *[21] Let M be a lattice module over a C-lattice L with 1_M is compact. Then for every proper element $X \in M$, there exists a maximal element N of M such that $X \leq N$.*

Let M be a lattice module over a multiplicative lattice L. The *Jacobson radical* $J(M) = \bigwedge Max(M)$ (see [3]).

Theorem 2.7. *Let M be a lattice module over a C-lattice L with 1_M is compact and $N \in M$. Then N is small if and only if $N \leq J(M)$.*

Proof. Suppose that N is a small element of M and $N \nleq J(M)$. Then there exists $K \in Max(M)$ such that $N \nleq K$ which implies $N \vee K = 1_M$. Since N is small, we have $K = 1_M$, a contradiction, and consequently, $N \leq J(M)$.
Conversely, suppose that $N \leq J(M)$ and N is non-small element of M. Then there exists a proper element K of M such that $N \vee K = 1_M$. Since M is a lattice module over a C-lattice L with 1_M is compact, by Lemma 2.6, there exists $P \in Max(M)$ such that $K \leq P$ therefore $1_M = N \vee K \leq N \vee P$ hence $N \vee P = 1_M$. But $N \leq J(M)$; therefore, $N \leq P$ and hence, $P = 1_M$, a contradiction, consequently, N is a small element of M.

Following definition is due to Eaman A. Al-Khouja [3].

Definition 2.8. A lattice module M over a multiplicative lattice L is said to be a local if it has only one maximal element.

A graph whose vertex set is empty is a *null graph* and a graph whose edge set is empty is an *empty graph* (see [18]).
Following Corollary follows from Theorem 2.7.

Corollary 2.9. *Let M be a lattice module over a C-lattice L with 1_M is compact. Then $G(M)$ is a null graph if and only if M is local.*

In the rest of paper we assume that $G(M)$ is a non-null graph.

Lemma 2.10. *Let M be a multiplication lattice module over a C-lattice L and $N, K \in M$. If P is a prime element of M with $N \wedge K \leq P$ then either $N \leq P$ or $K \leq P$.*

Proof. Suppose that P is a prime element of M with $N \wedge K \leq P$. Then by Lemma 2.3(2) and (3), $(N : 1_M) \wedge (K : 1_M) = (N \wedge K : 1_M) \leq (P : 1_M)$. Since P is prime, $(P : 1_M)$ is prime and hence quasi-prime element of L therefore $(N : 1_M) \wedge (K : 1_M) \leq (P : 1_M)$ implies either $(N : 1_M) \leq (P : 1_M)$ or $(K : 1_M) \leq (P : 1_M)$ and so either $(N : 1_M)1_M \leq (P : 1_M)1_M$ or $(K : 1_M)1_M \leq (P : 1_M)1_M$. Hence $N \leq P$ or $K \leq P$ because M is a multiplication lattice module.

Corollary 2.11. *Let M be a multiplication lattice module over a C-lattice L with $Max(M) = \{K_{i \in I}\}$ (I is an indexed set). Then for all non-empty proper finite subset Λ of I, $\wedge_{i\in\Lambda} K_i$ is non-small.*

Proof. Supoose that for a non-empty proper finite subset Λ of I, $\wedge_{i\in\Lambda} K_i$ is a small element of M. Then by Theorem 2.7, $\wedge_{i\in\Lambda} K_i \leq J(M) \leq K_j$ where $j \in I$ and $j \notin \Lambda$. Since every maximal is a prime element of M, by Lemma 2.10, for some $i \in \Lambda$, $K_i \leq K_j$, a contradiction, consequently, $\wedge_{i\in\Lambda} K_i$ is a non-small element of M.

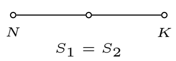
Theorem 2.12. *Let M be a lattice module over a C-lattice L with 1_M is compact. Then following statements are equivalent.*

(i) $G(M)$ is not connected.

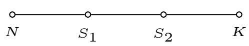
(ii) $|Max(M)| = 2$.

(iii) $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs.

Proof. 1) $\Rightarrow$ 2) Suppose that $|Max(M)| > 2$. Since $G(M)$ is not connected, we have components G_1 and G_2 of $G(M)$. Let $N, K \in M$ such that $N \in V(G_1)$ and $K \in V(G_2)$. It is clear that $N \wedge K$ is small. Since $|Max(M)| > 2$, by Lemma 2.6, there exist $S_1, S_2 \in Max(M)$ such that $N \leq S_1$ and $K \leq S_2$. Note that $N \wedge S_1$ and $K \wedge S_2$ are non-small becausse $(N \wedge S_1) \vee S_2 = 1_M$ and $(K \wedge S_2) \vee S_1 = 1_M$.
Case 1: If $S_1 = S_2$ then



is a path between G_1 and G_2, a contradiction.
Case 2: If $S_1 \neq S_2$. Since $|Max(M)| > 2$ then by Lemma 2.11, $S_1 \wedge S_2$ is non-small. Therefore



is a path, a contradiction, consequently, $|Max(M)| = 2$.
2) $\Rightarrow$ 3) Suppose that $Max(M) = \{S_1, S_2\}$ and $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$, $G_2(M)$ are subgraphs of $G(M)$. Since M is a lattice module over a C-lattice L with 1_M is compact and $Max(M) = \{S_1, S_2\}$, by lemma 2.6, $G_1 = \{N \in M | N \leq S_1$ and N is non-small$\}$ and $G_2 = \{X \in M | X \leq S_2$ and

X is non-small$\}$. To prove $G_i(M)$ (for $i = 1, 2$) is a complete subgraph, we contend that for any $N, K \in V(G_1(M))$, N, K are adjacent, i.e., $N \wedge K$ is non-small. Suppose that $N \wedge K$ is small. Then by Theorem 2.7, $N \wedge K \leq S_1 \wedge S_2 = J(M)$ therefore by Lemma 2.10, either $N \leq S_2$ or $K \leq S_2$ because every maximal is a prime element of M and $J(M) \leq S_2$. This implies either N or K is small, a contradiction to $N, K \in V(G_1(M))$, consequently, N and K are adjacent, i.e., $N \wedge K$ is non-small. Similarly, for any $X, Y \in V(G_2(M))$, $X \wedge Y$ is non-small. Hence $G_1(M)$ and $G_1(M)$ are complete subgraph.
Now to prove that $G_1(M)$ and $G_2(M)$ are disjoint, consider $N \in V(G_1(M))$ and $X \in V(G_2(M))$. If N and X are adjacent, then $N \wedge X$ is non-small. But $N \leq S_1$ and $X \leq S_2$ therefore $N \wedge X \leq S_1 \wedge S_2 = J(M)$ and so by Theorem 2.7, $N \wedge X$ is small, a contradiction, Consequently, $G_1(M)$ and $G_2(M)$ are disjoint subgraphs.
3) $\Rightarrow$ 1) Obvious.

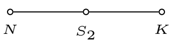
A *clique* of a graph is its maximal complete subgraph and the *clique number* of G is the number of vertices in the largest clique of a graph G, denoted by $\omega(G)$.
Following Corollary follows from Corollary 2.11 and Theorem 2.12.

Corollary 2.13. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact and $G(M)$ is a non-empty graph. Then $\omega(G(M)) \geq |Max(M)|$.*

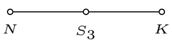
For a graph G, $d(a, b)$ is the length of the shortest path connecting a and b and if there is no path between a and b then $d(a, b) = \infty$. The diameter of a graph G is $diam(G) = \sup\{d(a, b) | a, b \in V(G)\}$.

Theorem 2.14. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact. If $G(M)$ is a connected graph then $diam(G(M)) \leq 2$.*

Proof. Suppose that $G(M)$ is a connected graph and $N, K \in V(G(M))$. If N, K is adjacent then $d(N, K) = 1$. Suppose that N, K are not adjacent vertices. Since M is a lattice module over a C-lattice L with 1_M is compact, by Lemma 2.6, there exist $S_1, S_2 \in Max(M)$ such that $N \leq S_1$ and $K \leq S_2$. If $N \wedge S_2$ is non-small then



is a path and so $d(N, K) = 2$. Similarly, if $K \wedge S_1$ is non-small then $d(N, K) = 2$. Now, suppose that both $N \wedge S_2$ and $K \wedge S_1$ are small. Then by Theorem 2.7, $N \wedge S_2 \leq J(M)$ and $K \wedge S_1 \leq J(M)$. Since $G(M)$ is a connected graph, by Theorem 2.12, $|Max(M)| > 2$ therefore there exists $S_3 \in Max(M)$ other than S_1, S_2 such that $N \wedge S_2 \leq J(M) \leq S_3$ and $N \wedge S_1 \leq J(M) \leq S_3$. Note that $S_1, S_2 \nleq S_3$, therefore by Lemma 2.10, $N \leq S_3$ and $K \leq S_3$ and hence there is a path

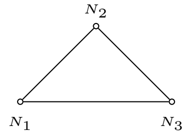


between N and K of length 2, consequently, $diam(G(M)) \leq 2$.

The *girth* of a graph G is the length of the shortest cycle in G and it is denoted by $g(G)$ (see [18]).

Theorem 2.15. *Let M be a multiplication lattice module over a C-lattice L with 1_M is compact. If $G(M)$ contains a cycle then $g(G(M)) = 3$.*

Proof. Suppose that $G(M)$ contains a cycle. Since $G(M)$ is not a null graph, we have $|Max(M)| \geq 2$.
Case-1: If $|Max(M)| = 2$, then by Theorem 2.12, $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs hence $g(G(M)) = 3$ because $G(M)$ contains a cycle.
Case-2: If $|Max(M)| \geq 3$. Then by Corollary 2.11, for $N_1, N_2, N_3 \in Max(M)$,

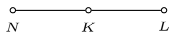


is a cycle in $G(M)$, consequently, $g(G(M)) = 3$.

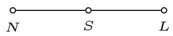
A vertex a in a connected graph G is a *cut vertex* if $G - \{a\}$ is disconnected.

Theorem 2.16. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact. If $G(M)$ is a connected graph then $G(M)$ has no cut vertex.*

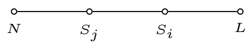
Proof. Suppose that $G(M)$ is a connected graph and $K \in V(G(M))$ is a cut vertex of $G(M)$. By definition $G(M) - \{K\}$ is not connected. This implies that for some $N, L \in V(G(M))$, K lies in each path between N and L. Since $G(M)$ is a connected graph, by Theorem 2.14, we have a path



of maximum length. Observe that, K is maximal because if K is not a maximal element of M then by Lemma 2.6, there exists $S \in Max(M)$ such that $K \leq S$ and so $K \wedge N \leq S \wedge N$. Since K and N are adjacent,i.e, $K \wedge N$ is non-small, by Remark 2.4, S and N are also adjacent vertices. Similarly, we can say that S and L are adjacent vertices and hence there is a path



which is contradiction to the fact K lies in each path between N and L, consequently K is maximal. Also, note that there exists a maximal element S_i other than K with $N \nleq S_i$, because if for each $S_i \in Max(M)$, $N \leq S_i$, then $N \wedge K \leq J(M)$, which implies by Theorem 2.7 that $N \wedge K$ is small, a contradiction. On the same line, we have $S_j \in Max(M)$ other than K with $L \nleq S_j$. Since $N \wedge L$ small, by Theorem 2.7 and Lemma 2.10, for distinct $S_i, S_j \in Max(M)$ other than K, if $N \nleq S_i$ and $L \nleq S_j$ then $N \leq S_j$ and $L \leq S_i$. Hence



is a path in $\Gamma(M) - \{K\}$, a contradiction, and consequently, $G(M)$ has no cut vertex.

Theorem 2.17. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact and $|Max(M)| < \infty$. Then there is no vertex in $V(G(M))$ which is adjacent to every other vertex.*

Proof. Suppose that there exists a vertex $N \in V(G(M))$ which is adjacent to every other vertex. Since 1_M is compact, by Lemma 2.6, there exists $S_i \in Max(M)$ such that $N \leq S_i$. By Corollary 2.11, for $S_j \in Max(M)$, $K = \wedge_{j \neq i} S_j$ is non-small. Since N is adjacent to every other vertex, $N \wedge K$ is non-small. But $N \wedge K \leq \wedge_{j \neq i} S_j \wedge S_i = J(M)$ therefore by Theorem 2.7, $N \wedge K$ is small, a contradiction, consequently, there is no vertex in $V(G(M))$ which is adjacent to every other vertex.

Following Corollary follows from Theorem 2.17.

Corollary 2.18. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact and $|Max(M)| < \infty$. Then $G(M)$ is not a complete graph.*

For $a \in V(G)$, the *neighborhood* of a is the set of vertices which are adjacent to a. A vertex of a graph G is said to be *pendant* if its neighborhood contains exactly one vertex.

Theorem 2.19. *Let M be a multiplication lattice module over a C-lattice L with 1_M is compact. If $G(M)$ has a pendant vertex then $|Max(M)| = 2$.*

Proof. Suppose that $N \in V(G(M))$ is a pendant vertex and $|Max(M)| \geq 3$. Then by Corollary 2.11, for distinct $S_i, S_j \in Max(M)$, $S_i \wedge S_j$ is non-small. This implies that for each S_i, $deg(S_i) \geq 2$ therefore $N \notin Max(M)$. Since 1_M is compact, by Lemma 2.6, there exists a maximal element S_1 of M such that $N \leq S_1$ and hence S_1 is the only element that is adjacent to N because N is pendant. Thus for $S_2 \in Max(M)$, $N \wedge S_2$ is small, therefore by Theorem 2.7, $N \wedge S_2 \leq J(M)$ and hence for $S_j \neq S_1, S_2$, $N \wedge S_2 \leq S_j$. Since S_j is prime and $S_2 \in Max(M)$, by Lemma 2.10, $N \leq S_j$, a contradiction, consequently, $|Max(M)| = 2$.

Corollary 2.20. *Let M be a multiplication lattice module over a C-lattice L with 1_M is compact. If $G(M)$ has a pendant vertex then $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are disjoint and complete subgraphs.*

Corollary 2.21. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact. Then $G(M)$ is not a star graph.*

Proof. Suppose that $G(M)$ is a star graph. Then by definition, $G(M)$ is connected and have a pendant vertex. Therefore by Theorem 2.12 and Theorem 2.19, $G(M)$ is not connected graph, a contradiction.

If the degree of each vertex is r then graph G is said to be *r-regular* graph.

Lemma 2.22. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact. If $G(M)$ is a $r-$regular graph, then $|Max(M)| = 2$ and $|V(G(M))| = 2r + 2$.*

Proof. Suppose that graph $G(M)$ is a $r-$regular and $|Max(M)| \geq 3$. By Lemma 2.5, $deg(S_1 \wedge S_2) \leq deg(S_1)$ where $S_1, S_2 \in Max(M)$. Note that $deg(S_1 \wedge S_2) \neq deg(S_1)$, because by Corollary 2.11, $\wedge S_{i \neq 2}$ is adjacent to S_1, however by Theorem 2.7, $\wedge S_{i \neq 2}$ is not adjacent to $S_1 \wedge S_2$. This implies that $deg(S_1 \wedge S_2) < deg(S_1)$ which is a contradiction to the fact that $G(M)$ is a $r-$regular graph and therefore $|Max(M)| \leq 2$. Since $G(M)$ is not a null graph, we have $|Max(M)| \neq 1$ hence $|Max(M)| = 2$. Let $Max(M) = \{N_1, N_2\}$. Then by Theorem 2.12, $G(M) = G_1(M) \cup G_2(M)$, where $G_1(M)$ and $G_2(M)$ are complete and disjoint subgraphs with $V(G_1) = \{S_i \in M | S_i \leq N_1\}$ and $V(G_2) = \{S_j | S_j \leq N_2\}$. Since $G(M)$ is a $r-$regular graph, $V(G_i) = r + 1$, consequently, $|V(G(M))| = 2r + 2$.

A subset $Z \subseteq V(G)$ is *independent*, if no two vertices in Z are adjacent. The independence number $\alpha(G)$ is the maximum size of an independent set in G.

Theorem 2.23. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact and $|Max(M)| < \infty$. Then $\alpha(G(M)) = |Max(M)|$.*

Proof. Suppose that $Max(M) = \{S_1, S_2, \cdots, S_n\}$. Then for $i = 1, 2 \cdots, n$, $Y = \{\wedge_{j=1, j \neq i}^{n} S_j\}$ is an independent set by Theorem 2.7. This implies that $\alpha(G(M)) \geq n$. Now to prove that $\alpha(G(M)) = n$, let $\alpha(G(M)) = m > n$ and let $X = \{N_1, N_2, \cdots, N_m\}$ be a maximal independent set. Note that for any $N_i \in X$, N_i is a non-small element of M, therefore for some $S_t \in Max(M)$, $N_i \nleq S_t$ by Theorem 2.7. Since $m > n$, by Pigeon hole principle, there exist $N_i, N_j \in X$ such that $N_i \nleq S_t$ and $N_j \nleq S_t$ where $1 \leq i, j \leq n$. But X is an independent set and $N_i, N_j \in X$ therefore $N_i \wedge N_j$ is small and so by Theorem 2.7, $N_i \wedge N_j \leq J(M) \leq S_t$. Since S_t is prime, by Lemma 2.10, either $N_i \leq S_t$ or $N_j \leq S_t$, a contradiction, consequently, $m = n = \alpha(\Gamma(M))$.

A *k-partite* graph is a graph whose vertices are or can be partitioned into k different independent sets. A *complete k-partite* graph is a k-partite graph in which there is an edge between every pair of vertices from different independent sets (see [18]).

Theorem 2.24. *Let M be a multiplication lattice module over a $C-$lattice L with 1_M is compact. Then $G(M)$ is not a complete n-partite graph.*

Proof. Suppose that $G(M)$ is a complete n-partite graph with independent sets $X_1, X_2, \cdots, X_n$. Note that, every complete graph is necessarily connected, therefore by Theorem 2.12, $|Max(M)| \geq 3$, and so by Corollary 2.11, for any $N_i, N_j \in Max(M)$, $N_i \wedge N_j$ is non-small. This implies that each X_i contains at most one maximal element, therefore by Pigeon hole principle $|Max(M)| \leq n$. Note that $|Max(M)| = n$, indeed if $|Max(M)| = l < n$ then there exists set X_t with no maximal element. By Corollary 2.11, for $N_j \in Max(M)$, $\wedge_{j\neq i} N_j$ is non-small. But $N_i \wedge (\wedge_{j\neq i} N_j) = J(M)$, therefore by Theorem 2.7, $\wedge_{j\neq i} N_j$ and N_i are non-adjacent vertices and hence $\wedge_{j\neq i} N_j \in X_i$. Now, suppose that $N \in X_t$. Since 1_M is compact, By Lemma 2.6, there exists a maximal element N_k such that $N \leq N_k$ and so N is adjacent to N_k. Note that $G(M)$ is a complete n-partite graph and $N_k \in X_k$ therefore N is adjacent to all vertices of X_k. Thus N is adjacent to $\wedge_{j\neq k} N_j$, i.e., $N \wedge (\wedge_{j\neq k} N_j)$ is non-small, which is contradiction to the fact that $N \wedge (\wedge_{j\neq k} N_j) \leq N_k \wedge (\wedge_{j\neq k} N_j) = J(M)$, consequently, $|Max(M)| = n$. Now, let $H = \wedge_{j=3}^{n} N_j$, where $N_j \in Max(M)$. Then by Corrolary 2.11, $H \in V(G(M))$. Note that $H \wedge N_1 = \wedge_{i\neq 2} N_i$ is non-small, i.e., H is adjacent to N_1. Similarly, H is adjacent to N_2. Hence $H \notin X_1$ and $H \notin X_2$. Also, by Theorem 2.7, for each i $(3 \leq i \leq n)$, $H \wedge N_i = H$ is non-small. This implies that H is adjacent to all maximal element N_i of M, hence for each i, $H \notin X_i$, a contradiction, consequently, $G(M)$ is not a complete n-partite graph.

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