UU Group Rings

P.V. Danchev\textsuperscript{1,*}, O. Al-Mallah\textsuperscript{2}

\textit{Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria}\textsuperscript{1}
\textit{Department of Mathematics and Statistics, King Faisal University, Saudi Arabia}\textsuperscript{2}

Abstract. A ring is called \textit{UU} if each its unit is a unipotent. We prove that the group ring $R[G]$ is a commutative UU ring if, and only if, $R$ is a commutative UU ring and $G$ is an abelian 2-group. This extends a result due to McGovern-Raja-Sharp (J. Algebra Appl., 2015) established for commutative nil-clean group rings. In some special cases we also discover when $R[G]$ is a non-commutative UU ring as our results are closely related to those obtained by Ko¸san-Wang-Zhou (J. Pure Appl. Algebra, 2016) and Sahinkaya-Tang-Zhou (J. Algebra Appl., 2017).

2010 Mathematics Subject Classifications: 20C07; 16U60; 16U99; 16S34; 20D15

Key Words and Phrases: UU rings, Nil-clean rings, Strongly nil-clean rings, Group rings

1. Introduction and Fundamentals

Throughout the present note our notations are in agreement with [6], [7] and [9]. For instance, as usual, the symbol $R[G]$ stands for the group ring of an arbitrary group $G$ over an arbitrary associative unital ring $R$, and $\omega(R[G])$ is its augmentation ideal. The used terminology is mainly standard as the new notions are stated explicitly below.

Imitating [3], we state the following:

\textbf{Definition 1.1.} A ring $R$ is said to be \textit{UU} if its unit group $U(R)$ satisfies the equality $U(R) = 1 + \text{Nil}(R)$, where $\text{Nil}(R)$ is the set of all nilpotent elements of $R$.

However, this definition is rather clumsy for applications and so the next necessary and sufficient condition from [3] will be useful in the sequel.

\textbf{Proposition 1.2.} A ring $R$ is \textit{UU} if, and only if, $2 \in \text{Nil}(R)$ and $U(R)$ is a 2-group.

On the other hand, a ring $R$ is called \textit{nil-clean} if $R = \text{Nil}(R) + \text{Id}(R)$, where $\text{Id}(R)$ is the set of all idempotents in $R$, that is, for every $r \in R$ there exist $q \in \text{Nil}(R)$ and $e \in \text{Id}(R)$ such that $r = q + e$. If, in addition, $qe = eq$ holds, the nil-clean ring is called \textit{strongly nil-clean}.

The following criterion, which was used in the proof of [8, Theorem 2.12], is independently proved in [3] and [5], respectively.

\textbf{Proposition 1.3.} A ring $R$ is strongly nil-clean if, and only if, the Jacobson radical $J(R)$ is nil and $R/J(R)$ is boolean.

*Corresponding author.

Email address: danchev@math.bas.bg; pvdanchev@yahoo.com * (Corresponding Author)

http://www.ibujournals.com 94 © 2018 EBM All rights reserved.
Even more, in [3] was showed that a ring is strongly nil-clean exactly when it is nil-clean $UU$ which amounts to a ring is strongly nil-clean uniquely when it is nil-clean and its unit group is a 2-group. In order to simplify the proof of Theorem 2.12 from [8], we shall use this key assertion in what follows (e.g., in Corollary 2.4) without any concrete referring.

A brief history of the best known principal achievements on group rings over such rings is like this: In [4] was found a complete description when the group ring $R[G]$ is nil-clean. This was further expanded in the non-commutative case in both [5] and [8] to the classes of strongly nil-clean and nil-clean rings, respectively.

The leitmotif of this short article is to generalize the aforementioned results to the large class of $UU$ rings as well as to give a more elementary and direct proof of a theorem from [5] (see [8, Theorem 2.12], too). It is worthwhile noticing that some partial statements on commutative group rings of $UU$ rings are given in [2, Section 5].

2. Main Results

We recall that a group is said to be locally finite if each its finite subset generates a finite subgroup, that is, each its finitely generated subgroup is finite. These groups are necessarily torsion. A type of such groups are the so-called locally normal groups, that are groups for which every finite subset can be embedded in a finite normal subgroup. For torsion abelian groups this property is always fulfilled, whereas in the non-abelian case the situation is more delicate being the classical Burnside’s problem solved in the negative.

Before proceed by proving our major assertion, we need the next pivotal instrument from [3].

- Let $I$ be a nil-ideal of a ring $R$. Then $R$ is $UU$ precisely when $R/I$ is $UU$.

Our chief statement is the following one:

**Theorem 2.1.** Let $G$ be a group and $R$ a ring.

(i) If $R[G]$ is $UU$, then $R$ is $UU$ and $G$ is a 2-group.

(ii) If $G$ is locally finite, then $R[G]$ is $UU$ if, and only if, $R$ is $UU$ and $G$ is a 2-group.

(iii) If $H$ is a normal subgroup of $G$ such that $H$ is locally normal and if $R[G]$ is $UU$, then $R[G/H]$ is $UU$.

**Proof.** (i) According to Proposition 1.2, we know that 2 is nilpotent in $R[G]$ and $U(R[G])$ is a 2-group. It now follows immediately that 2 is nilpotent in $R$ and that $U(R) \leq U(R[G])$ and $G \leq U(R[G])$ are both 2-groups. Again Proposition 1.2 applies to get that $R$ is $UU$, as wanted.

(ii) In view of point (i), we need to show only the "if" part. To that goal, since $G$ is locally finite, choosing $x \in \omega(R[G])$, we deduce that $x \in \omega(R[H])$ for some finite subgroup $H$ of $G$. But it is well known that the ideal $\omega(R[H])$ is nilpotent and thus nil (see, e.g., [1]). Hence the element $x$ is nilpotent, so that the ideal $\omega(R[G])$ is nil. Taking into account that $R[G]/\omega(R[G]) \cong R$ and the truthfulness of the bullet above, we are now done.

(iii) First of all, we observe that the following isomorphism of group rings

$$R[G]/(\omega(R[H]) \cdot R[G]) \cong R[G/H]$$

is fulfilled. We claim that the relative augmentation ideal $\omega(R[H]) \cdot R[G]$ of $R[G]$ is nil. In fact, since $H$ is locally normal, each element $z$ of this ideal is contained in the ideal $\omega(R[F]) \cdot R[G]$, where $F$ is a finite normal subgroup of $H$ and so normal in $G$. As above, it follows from [1, Theorem 9] that $\omega(R[F])$ is nilpotent whence so is $\omega(R[F]) \cdot R[G]$, because for any natural $i$ the formula $(\omega(R[F]) \cdot R[G])^i = (\omega(R[F]))^i \cdot R[G]$ holds, taking into account that $F$ is a normal subgroup of
$G$. That is why, $z$ is a nilpotent and so the claim sustained. Furthermore, the bullet alluded to above allows us to deduce that $R[G/H]$ is a UU ring, as pursued.

It is worth noticing that the claim in point (i) that $G$ is a 2-group contrasts the comments before Proposition 2.9 in [8].

The next affirmation is an immediate consequence of the preceding theorem.

**Corollary 2.2.** Suppose $R$ is a ring and $G$ is an abelian group. Then $R[G]$ is a UU ring if, and only if, $R$ is a UU ring and $G$ is a 2-group.

We now come to the promised above generalization of the basic result in [4].

**Corollary 2.3.** A group ring $R[G]$ is a commutative UU ring if, and only if, $R$ is a commutative UU ring and $G$ is an abelian 2-group.

So, we are ready to provide a more transparent proof of the following fact from [5] commented above:

**Corollary 2.4.** Suppose that $R$ is an arbitrary ring and $G$ is a locally finite group. Then $R[G]$ is strongly nil-clean if, and only if, $R$ is strongly nil-clean and $G$ is a 2-group.

**Proof.** The necessity is well-known and trivial. As for the sufficiency, it follows directly from Theorem 2.1 and [8, Theorem 2.3].

We close the work with the following question of interest.

**Problem 2.5.** Does it follow that Theorem 2.1 remains true without the assumption that $G$ is locally finite?

This query definitely will hold in the affirmative, provided that the next implication is valid: If $R$ is a ring having $\text{char}(R) = 2$ and $G$ is a finitely generated 2-group, then the augmentation ideal $\omega(R[G])$ of the group ring $R[G]$ is nil.

**Acknowledgement.** The authors would like to thank the referee for his/her careful reading of the manuscript and the valuable comments.

**References**


