On Weakly Contracted Petrov Symmetric Manifolds

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Abstract. In the domain of general relativity, Space-Matter Tensor, which was introduced by A. Z. Petrov, plays an important role to study and classify the general gravitational fields. This paper is concerned with the Contracted Petrov Tensor, which is found by taking contraction of Space-Matter Tensor. In the sequel, the notion of Weakly Contracted Petrov Symmetric Manifolds is introduced and studied. Further the existence of such type of manifolds is proved by citing an example.

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1. Introduction

For a general classification and study of all possible gravitational fields, A. Z. Petrov [9] introduced a tensor \( P \) of type \((0, 4)\) which satisfies all the algebraic properties of Riemannian curvature tensor. This particular tensor is known as space-matter tensor and is defined by

\[
P(X, Y, Z, U) = R(X, Y, Z, U) + \frac{k}{2} [g(X, U)T(Y, Z) + g(Y, Z)T(X, U) - g(X, Z)T(Y, U) - g(Y, U)T(X, Z)] - \sigma [g(X, U)g(Y, Z) - g(X, Z)g(Y, U)]
\]

for all \( X, Y, Z, U \in \chi(M^n) \), where \( \chi(M^n) \) is the module of all smooth vector fields over the manifold \( M^n \). Here \( R \) is the Riemannian curvature tensor of type \((0, 4)\), \( T \) is the energy-momentum tensor of type \((0, 2)\), \( k \) is a cosmological constant, \( \sigma \) is the energy density. Einstein’s field equation with cosmological constant is given by

\[
kT = S + \left( \lambda - \frac{r}{2} \right) g,
\]

where \( \lambda \) is a cosmological constant, \( r \) is the scalar curvature function of \( M \). By virtue of (9), (8) takes the form

\[
P(X, Y, Z, U)
\]

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\[ R(X, Y, Z, U) = g(X, U)S(Y, Z) + g(Y, Z)S(X, U) \]
\[ - g(X, Z)S(Y, U) - g(Y, U)S(X, Z) \]
\[ + \frac{1}{2} \left[ g(X, U)g(Y, Z) - g(Y, X)g(U, Z) \right] \]
\[ - \left( \sigma - \lambda + \frac{r}{2} \right) [g(X, U)g(Y, Z) - g(Y, X)g(U, Z)]. \]

Let \( \{ e_i : i = 1, 2, ..., n \} \) be an orthonormal frame of the tangent space at any point of the manifold \((M^n, g)\). We introduce a new tensor \( P_c \) and a scalar \( p \) such that
\[ P_c(X, Y) = g(QX, Y) = \sum_{i=1}^{n} P(e_i, X, Y, e_i) \quad \text{and} \quad p = \sum_{i=1}^{n} P(e_i, e_i), \]
where \( Q \) denotes the symmetric endomorphism corresponding to the tensor \( P_c \). The tensor \( P_c \) and the scalar \( p \) will be called 'contracted Petrov tensor' and 'Petrov scalar' respectively. By the virtue of (10) we have the expression for the contracted Petrov tensor as follows
\[ P_c(X, Y) = \frac{n}{2} S(X, Y) - \left[ (n - 1)(\sigma - \lambda) + (n - 2) \frac{r}{2} \right] g(X, Y). \]
(4)

And by (11) we have the Petrov scalar as follows
\[ p = n \left[ (n - 1)(\lambda - \sigma) - (n - 3) \frac{r}{2} \right]. \]
(5)

In this paper we introduce a new class of manifolds. For this, one requires a pre-discussion on weakly Ricci symmetric manifolds. An \( n \)-dimensional Riemannian manifold \((M^n, g)\) \((n > 2)\) is said to be \( n \)-dimensional weakly Ricci symmetric, denoted by \((WRS)_n\), if and only if its Ricci tensor \( S \) of type \((0, 2)\) is not identically zero and satisfies the following relation
\[ (\nabla_X S)(Y, Z) = \tilde{A}(X)S(Y, Z) + \tilde{B}(Y)S(X, Z) + \tilde{D}(Z)S(X, Y) \]
(6)
for all \( X, Y, Z \in \chi(M^n) \), where \( \chi(M^n) \) is as mentioned earlier and \( \tilde{A}, \tilde{B}, \tilde{D} \) are 1-forms (not simultaneously zero) and \( \nabla \) denotes the covariant differential operator with respect to the metric \( g \). This notion was introduced by Tamassy and Binh [11].

We now define a new type of Riemannian manifolds \((M^n, g)\) \((n > 3)\) whose contracted Petrov tensor \( P_c \) (non-vanishing identically) satisfies the condition
\[ (\nabla_X P_c)(Y, Z) = A(X)P_c(Y, Z) + B(Y)P_c(X, Z) + D(Z)P_c(X, Y) \]
(7)
for all \( X, Y, Z \in \chi(M^n) \) where \( A, B, D \) are 1-forms (not simultaneously zero) and \( \nabla \) denotes covariant differential operator with respect to the metric \( g \). Such a type of manifolds will be called 'weakly contracted Petrov symmetric manifolds' and denoted by \((WCPS)_n\).

Section 2 is concerned with some basic properties of the contracted Petrov tensor and weakly contracted Petrov symmetric manifold. It is shown that in a Riemannian manifold admitting Einstein’s field equation with cosmological constant, the contracted Petrov tensor will be of Codazzi type if and only if the Ricci tensor is of Codazzi type, provided the energy density is constant. After that it is also proved that a weakly contracted Petrov symmetric manifold admitting Einstein’s field equation with cosmological constant, is quasi-Einstein manifold, provided the Petrov scalar is...
non-zero and associated 1-forms $B \neq D$. At the end of this section it is shown that a Riemannian manifold admitting Einstein’s field equation with cosmological constant will be a weakly contracted Petrov symmetric manifold if and only if it is a weakly Ricci symmetric manifold, provided the scalar curvature and the energy density are connected by the relation 

$$r = \frac{2(n-1)}{n-2} (\lambda - \sigma).$$

The third section is devoted to study decomposable weakly contracted Petrov symmetric manifolds. It is proved that in a decomposable weakly contracted Petrov symmetric manifold, one of the decomposed manifolds is contracted Petrov flat and another is weakly contracted Petrov symmetric.

Section 4 is devoted to study weakly contracted Petrov symmetric manifolds admitting semi-symmetric metric connection and it is proved that if a weakly contracted Petrov symmetric manifold admits Einstein’s field equation with cosmological constant and semi-symmetric metric connection simultaneously with constant sectional curvature, then it is a manifold of quasi-constant curvature.

In the last section, an illustrative example of $(WCPS)_4$ admitting semi-symmetric metric connection is established and the theorem (5) is also verified.

For a general classification and study of all possible gravitational fields, A. Z. Petrov [9] introduced a tensor $P$ of type $(0, 4)$ which satisfies all the algebraic properties of Riemannian curvature tensor. This particular tensor is known as space-matter tensor and is defined by

$$P(X, Y, Z, U) = R(X, Y, Z, U) + \frac{k}{2} \left[ g(X, U)S(Y, Z) + g(Y, Z)S(X, U) - g(X, Z)S(Y, U) - g(Y, U)S(X, Z) \right] - \sigma \left[ g(X, U)g(Y, Z) - g(X, Z)g(Y, U) \right].$$

for all $X, Y, Z, U \in \chi(M^n)$, where $\chi(M^n)$ is the module of all smooth vector fields over the manifold $M^n$. Here $R$ is the Riemannian curvature tensor of type $(0, 4)$, $T$ is the energy-momentum tensor of type $(0, 2)$, $k$ is a cosmological constant, $\sigma$ is the energy density. Einstein’s field equation with cosmological constant is given by

$$kT = S + \left( \lambda + \frac{r}{2} \right) g,$$

where $\lambda$ is a cosmological constant, $r$ is the scalar curvature function of $M$. By virtue of (9), (8) takes the form

$$P(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{2} \left[ g(X, U)S(Y, Z) + g(Y, Z)S(X, U) - g(X, Z)S(Y, U) - g(Y, U)S(X, Z) \right] - \left( \sigma - \lambda + \frac{r}{2} \right) \left[ g(X, U)g(Y, Z) - g(X, Z)g(Y, U) \right].$$

Let $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal frame of the tangent space at any point of the manifold $(M^n, g)$. We introduce a new tensor $P_e$ and a scalar $p$ such that

$$P_e(X, Y) = g(QX, Y) = \sum_{i=1}^{n} P(e_i, X, Y, e_i) \quad \text{and} \quad p = \sum_{i=1}^{n} P_e(e_i, e_i),$$
where \( Q \) denotes the symmetric endomorphism corresponding to the tensor \( P_c \). The tensor \( P_c \) and the scalar \( p \) will be called ‘contracted Petrov tensor’ and ‘Petrov scalar’ respectively. By the virtue of (10) we have the expression for the contracted Petrov tensor as follows

\[
P_c(X, Y) = \frac{n}{2} S(X, Y) - \left[(n-1)(\sigma - \lambda) + (n-2) \frac{r}{2}\right] g(X, Y).
\] (11)

And by (11) we have the Petrov scalar as follows

\[
p = n \left[(n-1)(\lambda - \sigma) - (n-3) \frac{r}{2}\right].
\] (12)

In this paper we introduce a new class of manifolds. For this, one requires a pre-discussion on weakly Ricci symmetric manifolds. An \( n \)-dimensional Riemannian manifold \((M^n, g)\) \((n > 2)\) is said to be \( n \)-dimensional weakly Ricci symmetric, denoted by \((WRS)_n\) if and only if its Ricci tensor \( S \) of type \((0, 2)\) is not identically zero and satisfies the following relation

\[
(\nabla_X S)(Y, Z) = \tilde{A}(X)S(Y, Z) + \tilde{B}(Y)S(X, Z) + \tilde{D}(Z)S(X, Y)
\] (13)

for all \( X, Y, Z \in \chi(M^n) \), where \( \chi(M^n) \) is as mentioned earlier and \( \tilde{A}, \tilde{B}, \tilde{D} \) are 1-forms (not simultaneously zero) and \( \nabla \) denotes the covariant differential operator with respect to the metric \( g \). This notion was introduced by Tamassy and Binh [11].

We now define a new type of Riemannian manifolds \((M^n, g)(n > 3)\) whose contracted Petrov tensor \( P_c \) (non-vanishing identically) satisfies the condition

\[
(\nabla_X P_c)(Y, Z) = A(X)P_c(Y, Z) + B(Y)P_c(X, Z) + D(Z)P_c(X, Y)
\] (14)

for all \( X, Y, Z \in \chi(M^n) \) where \( A, B, D \) are 1-forms (not simultaneously zero) and \( \nabla \) denotes covariant differential operator with respect to the metric \( g \). Such a type of manifolds will be called ‘weakly contracted Petrov symmetric manifolds’ and denoted by \((WCPS)_n\).

Section 2 is concerned with some basic properties of the contracted Petrov tensor and weakly contracted Petrov symmetric manifold. It is shown that in a Riemannian manifold admitting Einstein’s field equation with cosmological constant, the contracted Petrov tensor will be of Codazzi type if and only if the Ricci tensor is of Codazzi type, provided the energy density is constant. After that it is also proved that a weakly contracted Petrov symmetric manifold admitting Einstein’s field equation with cosmological constant, is quasi-Einstein manifold, provided the Petrov scalar is non-zero and associated 1-forms \( B \neq D \). At the end of this section it is shown that a Riemannian manifold admitting Einstein’s field equation with cosmological constant will be a weakly contracted Petrov symmetric manifold if and only if it is a weakly Ricci symmetric manifold, provided the scalar curvature and the energy density are connected by the relation \( r = \frac{2(n-3)}{n-2} (\lambda - \sigma) \).

The third section is devoted to study decomposable weakly contracted Petrov symmetric manifolds. It is proved that in a decomposable weakly contracted Petrov symmetric manifold, one of the decomposed manifolds is contracted Petrov flat and another is weakly contracted Petrov symmetric.

Section 4 is devoted to study weakly contracted Petrov symmetric manifolds admitting semi-symmetric metric connection and it is proved that if a weakly contracted Petrov symmetric manifold admits Einstein’s field equation with cosmological constant and semi-symmetric metric connection simultaneously with constant sectional curvature, then it is a manifold of quasi-constant curvature.
In the last section, an illustrative example of \((WCPS)_{4}\) admitting semi-symmetric metric connection is established and the theorem (5) is also verified.

2. Basic Properties of Contracted Petrov Tensor and Weakly Contracted Petrov Symmetric Manifolds

This section deals with some elementary results on the contracted Petrov tensor and the weakly contracted Petrov symmetric manifolds.

**Proposition 1.** In a Riemannian manifold admitting Einstein’s field equation with cosmological constant, the contracted Petrov tensor with constant energy density will be of Codazzi type if and only if the Ricci tensor is of Codazzi type.

**Proof** Let us assume that in the Riemannian manifold under consideration the contracted Petrov tensor is of Codazzi type \([4]\). Then we have

\[
(\nabla_X P_c)(Y, Z) = (\nabla_Y P_c)(X, Z)
\]

for all smooth vector fields \(X, Y, Z\) on the manifold \(M\). In the view of (11) the above relation takes the form

\[
\frac{n}{2}(\nabla_X S)(Y, Z) - \left[ (n - 1)d\sigma(X) + \frac{n - 2}{2}dr(X) \right]g(Y, Z) = \frac{n}{2}(\nabla_Y S)(X, Z) - \left[ (n - 1)d\sigma(Y) + \frac{n - 2}{2}dr(Y) \right]g(X, Z).
\]

Further we consider that the energy density \(\sigma\) is constant. Then from the above equation we get

\[
n(\nabla_X S)(Y, Z) - (n - 2)dr(X)g(Y, Z) = n(\nabla_Y S)(X, Z) - (n - 2)dr(Y)g(X, Z).
\]

Contracting (15) with respect to \(Y\) and \(Z\) we get

\[
dr(X) = 0.
\]

Applying (16) in (15) we have

\[
\]

Therefore, the Ricci tensor is of Codazzi type. Now we consider that the relation (17) holds, i.e. the Ricci tensor is of Codazzi type. Then clearly \(dr(X) = 0\) for all \(X \in \chi(M^n)\). So with the help of the relation (11) we can easily verify that

\[
(\nabla_X P_c)(Y, Z) = (\nabla_Y P_c)(X, Z), \quad \text{whenever} \ \sigma \ \text{is a constant.}
\]

Therefore, the contracted Petrov tensor is of Codazzi type. Hence, our proposition is proved.

Now interchanging \(Y\) and \(Z\) in (14) we get

\[
\]
By the relation (14) and (18) we obtain
\[ \delta(Y)P_c(X, Z) = \delta(Z)P_c(X, Y), \tag{19} \]
where \( \delta(X) = B(X) - D(X) = g(X, \xi) \) for all \( X \in \chi(M^n) \) and \( \xi \) is the vector field associated to the 1-form \( \delta \). Contracting (19) with respect to \( X, Z \) we find
\[ p\delta(X) = \delta(QX). \tag{20} \]
In view of the equation (19) we have
\[ \delta(\xi)P_c(X, Y) = \delta(Y)P_c(X, \xi) = \delta(\xi)g(QX, \xi) = \delta(Y)\delta(QX), \]
which, by virtue of (20), yields
\[ P_c(X, Y) = pL(X)L(Y), \tag{21} \]
where \( L(X) = \frac{1}{\sqrt{\delta(\xi)}} \delta(X) \) for all \( X \), provided \( \delta(\xi) \neq 0 \), i.e. \( B \neq D \). So we have the following theorem:

**Theorem 2.** In a weakly contracted Petrov symmetric manifold admitting Einstein's field equation with cosmological constant, contracted Petrov tensor and Petrov scalar are connected by the relation (21).

**Definition 3.** A Riemannian manifold \((M^n, g)\) \((n > 3)\) is said to be quasi-Einstein [2], if its Ricci tensor \( S \) is not identically zero and there exists scalars \( \alpha_1, \alpha_2 \neq 0 \) such that
\[ S(X, Y) = \alpha_1 g(X, Y) + \alpha_2 \pi(X)\pi(Y) \]
for all \( X, Y \in \chi(M^n) \) and \( \pi \) is a non-zero 1-form.

By the virtue of (11) and (21) we get
\[ S(X, Y) = ag(X, Y) + bL(X)L(Y) \tag{22} \]
for all vector fields \( X \) and \( Y \) on \( M \), where \( a = \frac{2}{n} \left[ (n - 1)(\sigma - \lambda) + (n - 2)\frac{\lambda}{2} \right] \) and \( b = \frac{2}{n}p \). Therefore, we get the following theorem:

**Theorem 4.** In a weakly contracted Petrov symmetric manifold admitting Einstein's field equation with cosmological constant and associated non-zero 1-forms \( B \neq D \), if the Petrov scalar is non zero then it will be a quasi-Einstein manifold, otherwise it reduces to an Einstein manifold.

Now let us consider that in the manifold \( M \) the relation (14) holds and \( r = \frac{2(n-1)}{n-2}(\lambda - \sigma) \). Then using the equation (11) we have
\[ (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(X, Y). \tag{23} \]
Conversely, we consider that in \( M \) the relation (23) holds and \( r = \frac{2(n-1)}{n-2}(\lambda - \sigma) \), then from the equation (11) we have
\[ (\nabla_X P_c)(Y, Z) = A(X)P_c(Y, Z) + B(Y)P_c(X, Z) + D(Z)P_c(X, Y). \]
Hence, we have the following theorem:

**Theorem 5.** The necessary and sufficient condition for a Riemannian manifold admitting Einstein's field equation with cosmological constant to be a weakly contracted Petrov symmetric manifold is that it is weakly Ricci symmetric, provided the scalar curvature and the energy density are connected by the relation \( r = \frac{2(n-1)}{n-2}(\lambda - \sigma) \).
3. Decomposable Weakly Contracted Petrov Symmetric Manifolds

This section is concerned with decomposable weakly contracted Petrov symmetric manifolds. A Riemannian manifold \( (M^n, g) \) \((n > 2)\) is said to be decomposable or product manifold if it can be expressed as \( M_1^l \times M_2^{n-l} \) for \( 1 < l < n − 1 \).

We assume that the \((WCPS)_n\) under consideration is a decomposable manifold with \( M_1^l \) and \( M_2^{n-l} \) as its decomposed manifolds, i.e, \( M^n = M_1^l \times M_2^{n-l} \) where \( 1 < l < n − 1 \). Now for \( \tilde{X}, \tilde{Y}, \tilde{Z}, X, Y, Z \in \chi(M^n) \) where \( \tilde{X}, \tilde{Y}, \tilde{Z} \in \chi(M_1) \) and \( X, Y, Z \in \chi(M_2) \) by (14) we have

\[
(\nabla_{\tilde{X}} P_c)(\tilde{Y}, \tilde{Z}) = A(\tilde{X})P_c(\tilde{Y}, \tilde{Z}) + B(\tilde{Y})P_c(\tilde{X}, \tilde{Z}) + D(\tilde{Z})P_c(\tilde{X}, \tilde{Y}),
\]

which implies

\[ A(\tilde{X})P_c(\tilde{Y}, \tilde{Z}) = 0, \tag{25} \]

and

\[ (\nabla_X P_c)(Y, Z) = A(X)P_c(Y, Z) + B(Y)P_c(X, Z) + D(Z)P_c(X, Y), \tag{26} \]

which implies

\[ A(X)P_c(Y, Z) = 0. \tag{27} \]

Similarly we can show that

\[ B(X)P_c(Y, Z) = 0; \quad B(\tilde{X})P_c(\tilde{Y}, \tilde{Z}) = 0, \tag{28} \]

as well as

\[ D(X)P_c(Y, Z) = 0; \quad D(\tilde{X})P_c(\tilde{Y}, \tilde{Z}) = 0. \tag{29} \]

Now by the definition of \((WCPS)_n\), \( A(X), B(X), D(X) \) are not simultaneously zero. Let \( A(\tilde{X}) \) be non zero for any non-zero \( \tilde{X} \in \chi(M_2) \), therefore from (25) we have \( P_c(\tilde{Y}, \tilde{Z}) = 0, \forall \tilde{Y}, \tilde{Z} \in \chi(M_1) \) which means that \( M_1^l \) is contracted Petrov flat in nature. If \( A(\tilde{X}) \) is non zero for any non-zero \( \tilde{X} \in \chi(M_1) \), then from the relation (27) we get \( P_c(\tilde{Y}, \tilde{Z}) = 0 \) for all \( \tilde{Y}, \tilde{Z} \in \chi(M_2) \) which means \( M_2^{n-l} \) is contracted Petrov flat. Similarly for nonzero \( B(\tilde{X}), D(\tilde{X}) \) and \( B(X), D(X) \) we can show that \( M_1^l \) and \( M_2^{n-l} \) both are contracted Petrov flat.

Again by our hypothesis the \((WCPS)_n\) under consideration is a decomposable manifold. So \( M_1, M_2 \) both can not be simultaneously contracted Petrov flat. Let us consider \( M_1 \) as contracted Petrov flat, then \( M_2 \) is not contracted Petrov flat, i.e, from (27), (28), (29) we get \( A(\tilde{X}), B(\tilde{X}), D(\tilde{X}) \) are zero for all \( \tilde{X} \in \chi(M_1) \). But \( A(X), B(X), D(X) \) are not simultaneously zero for all non-zero \( X \in \chi(M^n) \). Therefore, \( A(X), B(X), D(X) \) are not simultaneously zero for all non-zero \( X \in \chi(M_2) \). Hence, by (14) we have

\[
(\nabla_X P_c)(Y, Z) = A(X)P_c(Y, Z) + B(Y)P_c(X, Z) + D(Z)P_c(X, Y), \tag{30} \]

which means \( M_2 \) is \((WCPS)_{n-1}\). Hence, we can state the following theorem:

**Theorem 6.** In a decomposable weakly contracted Petrov symmetric manifold, one of the decomposed manifolds is contracted Petrov flat and the other is weakly contracted Petrov symmetric.
4. Weakly Contracted Petrov Symmetric Manifolds Admitting a Semi-symmetric Metric Connection

This section is concerned with weakly contracted Petrov symmetric manifolds admitting a semi-symmetric metric connection. In 1924 Friedmann and Schouten [5] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932 Hayden [6] introduced the idea of metric connection with torsion on a Riemannian manifold. After that, in 1970 systematic study of the semi-symmetric metric connection on a Riemannian manifold was initiated by K. Yano [12].

A smooth linear connection $\tilde{\nabla}$ on an $n$-dimensional Riemannian manifold $(M^n, g)$, with the Riemannian connection $\nabla$, is said to be semi-symmetric [5] if and only if there exists a non-zero 1-form $\omega(X) = g(X, \rho)$ such that the torsion tensor $\tau$ of the connection $\tilde{\nabla}$ satisfies

$$\tau(X, Y) = \omega(Y)X - \omega(X)Y$$

for all $X, Y \in \chi(M^n)$ and $\rho$ being the vector field associated with the 1-form $\omega$. The 1-form $\omega$ is called the associated 1-form of the semi-symmetric connection and the vector field $\rho$ is called the associated vector field of the connection. A semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection [6] if it satisfies

$$\tilde{\nabla}g = 0,$$

in addition to the previous relation. The relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Riemannian connection $\nabla$ of $(M^n, g)$ [12] is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)\rho.$$  \hspace{1cm} (31)

In particular, if we let the 1-form $\omega$ to vanish identically then a semi-symmetric metric connection reduces to the Riemannian connection (at least for calculation purpose). The covariant differentiation of the 1-form $\omega$ with respect to $\tilde{\nabla}$ [12] is given by

$$(\tilde{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) + \omega(X)\omega(Y) - \omega(\rho)g(X, Y).$$

and the relation between the curvature tensors $R$ and $\tilde{R}$ of the manifold with respect to Riemannian connection and semi-symmetric metric connection respectively [12], is

$$\tilde{R}(X, Y)Z = R(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)LX + g(X, Z)LY,$$

where $\alpha$ is a tensor field of type $(0, 2)$ given by

$$\alpha(X, Y) = g(LX, Y) = (\nabla_X \omega)(Y) - \omega(X)\omega(Y) + \frac{1}{2} \omega(\rho)g(X, Y)$$

for any vector fields $X$ and $Y$. From (32) it follows that

$$S(Y, Z) = \tilde{S}(Y, Z) + (n - 2)\alpha(Y, Z) - ag(Y, Z),$$

where $\tilde{S}$ denotes the Ricci tensor with respect to $\tilde{\nabla}$, $a = \text{trace} \, \alpha$. Contracting (34) with respect to $Y$ and $Z$, it can be easily found that

$$\tilde{r} = r - 2(n - 1)a,$$

for any vector fields $X$ and $Y$. From (32) it follows that
where \( \tilde{r} \) denotes the scalar curvature of the manifold with respect to \( \tilde{\nabla} \).

A Riemannian manifold \((M^n, g) \) \((n > 2)\) admitting a semi-symmetric metric connection \( \tilde{\nabla} \) is said to be weakly Ricci symmetric if and only if for all \( X, Y, Z \in \chi(M^n) \) its Ricci tensor \( \tilde{S} \) is not identically zero and satisfies the following relation

\[
(\tilde{\nabla}_X \tilde{S})(Y, Z) = \tilde{A}(X)\tilde{S}(Y, Z) + \tilde{B}(Y)\tilde{S}(X, Z) + \tilde{D}(Z)\tilde{S}(X, Y),
\]

where \( \tilde{A}, \tilde{B}, \tilde{D} \) are non-zero 1-forms (not simultaneously zero). Such an \( n \)-dimensional manifold is denoted by \([(WRS)_n, \tilde{\nabla}]\).

The sectional curvature \( K(\Pi) \) of a tangent plane \( \Pi \) to the Riemannian manifold \( M \) with Levi-Civita connection is defined by

\[
K(X, Y) = \frac{g(X, X)g(Y, Y) - g(X, Y)^2}{g(X, X)g(Y, Y) - g(X, Y)^2} = g(R(X, Y)Y, X)
\]

for all \( X, Y \in \chi(M^n) \) and the sectional curvature \( \tilde{K}(\Pi) \) of a tangent plane \( \Pi \) to the Riemannian manifold \( M \) admitting a semi-symmetric metric connection \( \tilde{\nabla} \) is defined in the same fashion by

\[
\tilde{K}(X, Y) = \frac{g(X, X)g(Y, Y) - g(X, Y)^2}{g(X, X)g(Y, Y) - g(X, Y)^2} = g(\tilde{R}(X, Y)Y, X)
\]

for all \( X, Y \in \chi(M^n) \). The conformal curvature tensor \( C \) is defined by

\[
C(X, Y, U, V) = R(X, Y, U, V) - \frac{1}{n - 2} (g \wedge S)(X, Y, U, V)
\]

\[
- \frac{r}{(n - 1)(n - 2)} G(X, Y, U, V),
\]

where \( g \wedge S \) is Kulkarni-Nomizu product of \( g \) and \( S \). According to Yano [12], a Riemannian manifold admitting a semi-symmetric metric connection with constant sectional curvature is conformally flat. Let \( \tilde{K}(\Pi) \) be constant then \( C = 0 \). This implies

\[
R(X, Y, U, V) = \frac{1}{n - 2} (g \wedge S)(X, Y, U, V)
\]

\[
+ \frac{r}{(n - 1)(n - 2)} G(X, Y, U, V).
\]

The above relation is trivial for \( n = 3 \). In the view of (11) and (21) the above equation takes the following form

\[
R(X, Y, U, V) = \beta_1 [g(X, V)L(Y)L(U) + g(Y, U)L(X)L(V)]
\]

\[
- g(X, U)L(Y)L(V) - g(Y, V)L(X)L(U)]
\]

\[
+ \beta_2 [g(X, V)g(Y, U) - g(X, U)g(Y, V)],
\]

where \( \beta_1 = \frac{2p}{n(n - 2)}, \beta_2 = \frac{4(n - 1)(\sigma - \lambda)}{n(n - 2)} + \frac{2n^2 - 5n + 4}{n(n - 1)(n - 2)} r \). So we have the following theorem:

**Theorem 7.** A weakly contracted Petrov symmetric manifold admitting Einstein’s field equation with cosmological constant and admitting a semi-symmetric metric connection \( \tilde{\nabla} \) with constant sectional curvature, is a manifold of quasi-constant curvature.
Replacing $V$ by $\xi$ in (39) we get

$$L(R(X, Y)U) = (\beta_1 + \beta_2)[L(X)g(Y, U) - L(Y)g(X, U)].$$

Let us consider that the $(WCPS)_n$ under consideration is Ricci semi-symmetric, then we have

$$R(X, Y) \cdot S(U, V) = 0$$

for all $X, Y, U, V \in \chi(M^n)$. By the help of (22), the above relation is assumed in the following form

$$L(U)L(R(X, Y)V) + L(V)L(R(X, Y)U) = 0.$$  

Applying (40) in (41) we have

$$((\beta_1 + \beta_2)[L(U)[L(X)g(Y, V) - L(Y)g(X, V)]$$

$$+ L(V)[L(X)g(Y, U) - L(Y)g(X, U)]) = 0.$$  

Putting $U = \xi$ in (42) we obtain

$$\beta_1 + \beta_2 = 0,$$

which implies

$$2p + 4(n - 1)(\sigma - \lambda) + \frac{2n^2 - 5n + 4}{n - 1}r = 0.$$  

In view of the relation (12), the above relation takes the following form

$$r = 2(n - 1)^2(n - 2) - 6n^2 + 8n - 4(\lambda - \sigma).$$

Therefore, we get the following theorem:

**Theorem 8.** If a weakly contracted Petrov symmetric manifold admitting Einstein's field equation with cosmological constant and a semi-symmetric metric connection with constant sectional curvature is Ricci semi-symmetric, then its scalar curvature satisfies the relation (43).

### 5. Example of Weakly Contracted Petrov Symmetric Manifolds

This section is devoted to cite an example of a weakly Ricci symmetric manifold, which is also a weakly contracted Petrov symmetric manifold.

**Example:-** Let us consider a manifold $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1 \neq 0\}$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \Gamma^4_{14} = \Gamma^4_{41} = -\Gamma^4_{22} = -\Gamma^4_{33} = -\Gamma^4_{44}.$$  

Using the above relations, we can find the non-vanishing components of the Ricci tensor and their covariant derivatives are

$$\begin{cases}
R_{22} = \frac{1}{2} = R_{33} = R_{44}, & R_{22,1} = -\frac{1}{2} = R_{33,1} = R_{44,1}, \\
R_{12,2} = -\frac{1}{2} = R_{21,2} = R_{31,3} = R_{31,3} = R_{14,4} = R_{41,4}.
\end{cases}$$
where ',' denotes the covariant differentiation with respect to the Levi-Civita connection \( \nabla \) and \( R_{ij} \) denote the components of the Ricci tensor \( S \) in terms of local coordinate system. It can be easily shown that the scalar curvature of the manifold is \( r = \frac{3}{2}e^{-x^1} \), which is non-vanishing and non-constant. Let us consider the associated 1-forms \( \hat{A}, \hat{B}, \hat{D} \) as follows:

\[
\begin{align*}
\hat{A}_i &= \begin{cases} 
-1 & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases} \\
\hat{B}_i &= \begin{cases} 
-\frac{1}{2} & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases} \\
\hat{D}_i &= \begin{cases} 
-\frac{1}{2} & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

(46)

where \( \hat{A}_i, \hat{B}_i, \) and \( \hat{D}_i \) are the components of the 1-forms \( \hat{A}, \hat{B}, \) and \( \hat{D} \) respectively in terms of local coordinates system. In terms of local coordinate system, (13) can be written as

\[
R_{ij,k} = \hat{A}_k R_{ij} + \hat{B}_i R_{kj} + \hat{D}_j R_{ki}, \quad (i, j, k = 1, 2, 3, 4). \tag{47}
\]

By virtue of (45) and (46) we can show that

the right hand side of (47) = \( \hat{A}_1 R_{22} + \hat{B}_2 R_{12} + \hat{D}_2 R_{12} \)

= \( -\frac{1}{2} \)

= \( R_{22,1} \)

= left hand side of (47),

for \( i = 2 = j \) and \( k = 1 \). In a similar manner it can be shown that the relation (47) is true for other values of \( i, j \) and \( k \). Therefore, \((M^4, g)\) under consideration is a \((WRS)_4\).

Further, we assume that \( \sigma = \lambda - \frac{1}{2}e^{-x^1} \). Thus, we have

\[
r = 3(\lambda - \sigma)
\]

and the non-vanishing components of the space-matter tensor \( P \) are

\[
P_{2332} = \frac{e^{x^1}}{2} = P_{2442} = P_{3443} \tag{48}
\]

By virtue of (44) and (48) we have the non-vanishing components of the contracted Petrov tensor and their covariant derivatives are

\[
\begin{align*}
P_{e22} &= 1 = P_{e33} = P_{e44}, \\
P_{e21,2} &= -\frac{3}{2} = P_{e21,2} = P_{e33,3} = P_{e31,3} = P_{e43,4} = P_{e41,4},
\end{align*}
\]

(49)

where ',' denotes the covariant differentiation with respect to the Levi-Civita connection \( \nabla \). Let us consider the 1-forms \( A, B, D \) as follows:

\[
\begin{align*}
A_i &= \begin{cases} 
-1 & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases} \\
B_i &= \begin{cases} 
-\frac{1}{2} & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases} \\
D_i &= \begin{cases} 
-\frac{1}{2} & \text{for } i = 1, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

(50)
where $A_i$, $B_i$ and $D_i$ are the components of the 1-forms $A$, $B$ and $D$ respectively in terms of local coordinates system. In terms of local coordinate system, (14) can be written as

$$P_{cij,k} = A_k P_{cij} + B_i P_{ckj} + D_j P_{cki}, \quad (i, j, k = 1, 2, 3, 4). \tag{51}$$

By virtue of (49) and (50) we can show that

right hand side of (51) = $A_1 P_{c22} + B_2 P_{c12} + D_2 P_{c12} = -1 = P_{c22,1} = \text{left hand side of (51)},$

for $i = 2 = j$ and $k = 1$. In a similar manner it can be shown that the relation (51) is true for other values of $i$, $j$ and $k$. Therefore, $(M^4, g)$ under consideration is also a $(\text{WCPS})_4$ and thus, Theorem 5 is verified.

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References


