New refinements of two well-known inequalities

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Abstract. In this note, we improve the useful inequalities \( \sin(x) \sinh(x) \leq x^2 \) and \( \cos(x) \cosh(x) \leq 1 \) by the use of infinite products.

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1. Introduction

During the past several years, sharp inequalities involving trigonometric and hyperbolic functions have received a lot of attention. Thanks to their usefulness in all areas of mathematics. Old and new such inequalities, as well as refinements of the so-called Jordan’s, Cusa-Huygens and Wilker inequalities, can be found in [2, 3, 4, 5, 6, 9, 8, 11, 10], and the references therein.

In this note, we focus our attention on the following famous results proved by [8, Lemma 1 page 67] and [10, Lemma 1 page 148].

Lemma 1.1. [8, Lemma 1], [10, Lemma 1] For \( x \in (0, \pi/2) \), we have
\[
\sin(x) \sinh(x) \leq x^2, \quad \cos(x) \cosh(x) \leq 1.
\]

The proof of Lemma 1.1 is based on the studies of appropriate functions. The first inequality is also known in the form \( \sin(x)/x \leq x/\sinh(x) \) or \( \sin(x) \leq x^2/\sinh(x) \). In this note, we develop refinements of the two inequalities in Lemma 1.1 by using infinite products and applying the methodology developed in [1]. Sharp lower bounds of various nature are also established, with graphical supports.

2. Main results

In the following result, we propose a double inequality for \( \sin(x) \sinh(x) \), implying \( \sin(x) \sinh(x) \leq x^2 \) for the upper bound.

Proposition 2.1. For \( x \in (0, \alpha) \) and \( \alpha \in (0, \pi) \), we have
\[
x^2 \exp(-\beta x^4) \leq \sin(x) \sinh(x) \leq x^2 \exp \left(-\frac{x^4}{90}\right),
\]
with \( \beta = -\ln(\sin(\alpha) \sinh(\alpha)/\alpha^2)/\alpha^4 \).

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**Proof.** We have the following infinite products:

\[
\frac{\sin(x)}{x} = \prod_{k=1}^{+\infty} \left( 1 - \frac{x^2}{\pi^2 k^2} \right), \quad \frac{\sinh(x)}{x} = \prod_{k=1}^{+\infty} \left( 1 + \frac{x^2}{\pi^2 k^2} \right).
\]

Therefore, using the inequality \(1 + y \leq e^y\) for \(y \in \mathbb{R}\) and \(\zeta(4) = \sum_{k=1}^{+\infty} (1/k^4) = \pi^4/90\), we have

\[
\sin(x)\sinh(x) = x^2 \prod_{k=1}^{+\infty} \left( 1 - \frac{x^4}{\pi^4 k^4} \right) \geq x^2 \prod_{k=1}^{+\infty} \exp \left(- \frac{x^4}{\pi^4 k^4} \right)
\]

\[
= x^2 \exp \left(-\frac{x^4}{\pi^4 \zeta(4)} \right) = x^2 \exp \left(-\frac{x^4}{90} \right).
\]

For the lower bound, using the Bernoulli inequality, i.e. for \(u, v \in (0, 1)\), we have \(1 - uv \geq (1 - v)^u\), we obtain, for \(x \in (0, \alpha)\),

\[
\sin(x)\sinh(x) = x^2 \prod_{k=1}^{+\infty} \left( 1 - \frac{x^4}{\alpha^4 \pi^4 k^4} \right) \geq x^2 \prod_{k=1}^{+\infty} \left( 1 - \frac{\alpha^4}{\pi^4 k^4} \right)^{x^4/\alpha^4}
\]

\[
= x^2 \left[ \prod_{k=1}^{+\infty} \left( 1 - \frac{\alpha^4}{\pi^4 k^4} \right) \right]^{x^4/\alpha^4} = x^2 \left[ \frac{\sin(\alpha)\sinh(\alpha)}{\alpha^4} \right]^{x^4/\alpha^4}
\]

\[
= x^2 \exp(-\beta x^4).
\]

This ends the proof of Proposition 2.1.

The proposition below proposes a tight polynomial lower bound for \(\sin(x)\sinh(x)\).

**Proposition 2.2.** For \(x \in (0, \pi)\), we have

\[
x^2 \left( 1 - \frac{x^4}{\pi^4} \right)^{\pi^4/90} \leq \sin(x)\sinh(x).
\]

**Proof.** By using the Bernoulli inequality, i.e. for \(u, v \in (0, 1)\), we have \(1 - uv \geq (1 - v)^u\), and \(\zeta(4) = \pi^4/90\), we have

\[
\sin(x)\sinh(x) = x^2 \prod_{k=1}^{+\infty} \left( 1 - \frac{x^4}{\pi^4 k^4} \right) \geq x^2 \prod_{k=1}^{+\infty} \left( 1 - \frac{x^4}{\pi^4} \right)^{1/k^4}
\]

\[
= x^2 \left( 1 - \frac{x^4}{\pi^4} \right)^{\zeta(4)} = x^2 \left( 1 - \frac{x^4}{\pi^4} \right)^{\pi^4/90}.
\]

This ends the proof of Proposition 2.2.

Figure 1 illustrates the sharpness of the bounds in Propositions 2.1 and 2.2 for \(x \in (0, 3)\). We see that these inequalities are sharp, particularly for \(x \in (0, 1.5)\). At least for \(x \in (1.5, 1)\), the lower bound in Proposition 2.2 is clearly more sharp than the one in Proposition 2.1.
Similarly, the following result determines a double inequality for $\cos(x)\cosh(x)$, implying $\cos(x)\cosh(x) \leq 1$ for the upper bound.

**Proposition 2.3.** For $x \in (0, \alpha)$ and $\alpha \in (0, \pi/2)$, we have

\[
\exp(-\gamma x^4) \leq \cos(x)\cosh(x) \leq \exp\left(-\frac{x^4}{6}\right),
\]

with $\gamma = -\ln(\cos(\alpha)\cosh(\alpha))/\alpha^4$.

**Proof.** We have the following infinite products:

\[
\cos(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right), \quad \cosh(x) = \prod_{k=1}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right).
\]

Therefore, using the inequality $1+y \leq e^y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty} (1/(2k-1)^4) = (15/16)\zeta(4) = (15/16)(\pi^4/90) = (1/16)(\pi^4/6)$, we have

\[
\cos(x)\cosh(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right) \prod_{k=1}^{+\infty} \left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right)
\leq \prod_{k=1}^{+\infty} \exp\left(-\frac{16x^4}{\pi^4(2k-1)^4}\right)
= \exp\left(-x^4\frac{16}{\pi^4}\sum_{k=1}^{+\infty} \frac{1}{(2k-1)^4}\right) = \exp\left(-\frac{x^4}{6}\right).
\]

By virtue of the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geq (1 - v)^u$, we obtain, for $x \in (0, \alpha)$,

\[
\cos(x)\cosh(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{x^4}{\alpha^4 \pi^4(2k-1)^4}\right) \geq \prod_{k=1}^{+\infty} \left(1 - \frac{16\alpha^4}{\pi^2(2k-1)^4}\right)^{x^4/\alpha^4}
\]
\[
\prod_{k=1}^{+\infty} \left( 1 - \frac{16\alpha^4}{\pi^4(2k-1)^4} \right)^{x^4/\alpha^4} = \cos(\alpha)\cosh(\alpha)]^{x^4/\alpha^4} = \exp(-\gamma x^4).
\]

The proof of Proposition 2.3 is completed.

The proposition below presents a sharp polynomial lower bound for \(\cos(x)\cosh(x)\).

**Proposition 2.4.** For \(x \in (0, \pi/2)\), we have

\[
\left(1 - \frac{16x^4}{\pi^4}\right)^{\pi^4/96} \leq \cos(x)\cosh(x).
\]

**Proof.** By using the Bernoulli inequality, i.e. for \(u, v \in (0, 1)\), we have \(1 - uv \geq (1 - v)^u\), and

\[
+\infty \sum_{k=1} \left( \frac{1}{2k-1} \right)^4 = \pi^4/96,
\]

we have

\[
\cos(x)\cosh(x) = \prod_{k=1}^{+\infty} \left( 1 - \frac{16x^4}{\pi^4(2k-1)^4} \right) \geq \prod_{k=1}^{+\infty} \left( 1 - \frac{16x^4}{\pi^4} \right)^{1/(2k-1)^4}
= \left( 1 - \frac{16x^4}{\pi^4} \right)^{+\infty \sum_{k=1} (1/(2k-1)^4)} = \left( 1 - \frac{16x^4}{\pi^4} \right)^{\pi^4/96}.
\]

This ends the proof of Proposition 2.4.

Figure 2 illustrates the sharpness of the bounds in Propositions 2.3 and 2.4 for \(x \in (0, 1.5)\). At least for \(x \in (0.5, 1.5)\), we see that the lower bound in Proposition 2.4 is clearly more sharp to the one in Proposition 2.3.

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References


