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New refinements of two well-known inequalities

C. Chesneau[1,*], Y. J. Bagul[2]

LMNO, University of Caen-Normandie, Caen, 14032, France[1]
Department of Mathematics, K. K. M. College Manwath, Dist: Parbhani(M.S.) - 431505, India[2]

Abstract. In this note, we improve the useful inequalities $sin(x)sinh(x) \leqslant x^2$ and $cos(x)cosh(x) \leqslant 1$ by the use of infinite products.
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1. Introduction

During the past several years, sharp inequalities involving trigonometric and hyperbolic functions have received a lot of attention. Thanks to their usefulness in all areas of mathematics. Old and new such inequalities, as well as refinements of the so-called Jordan's, Cusa-Huygens and Wilker inequalities, can be found in [2, 3, 4, 5, 7, 6, 9, 8, 11, 10], and the references therein.

In this note, we focus our attention on the following famous results proved by [8, Lemma 1 page 67] and [10, Lemma 1 page 148].

Lemma 1.1. *[8, Lemma 1], [10, Lemma 1] For $x \in (0, \pi/2)$, we have*

$$sin(x)sinh(x) \leqslant x^2, \quad cos(x)cosh(x) \leqslant 1.$$

The proof of Lemma 1.1 is based on the studies of appropriate functions. The first inequality is also known in the form $sin(x)/x \leqslant x/sinh(x)$ or $sin(x) \leqslant x^2/sinh(x)$. In this note, we develop refinements of the two inequalities in Lemma 1.1 by using infinite products and applying the methodology developed in [1]. Sharp lower bounds of various nature are also established, with graphical supports.

2. Main results

In the following result, we propose a double inequality for $sin(x)sinh(x)$, implying $sin(x)sinh(x) \leqslant x^2$ for the upper bound.

Proposition 2.1. *For $x \in (0, \alpha)$ and $\alpha \in (0, \pi)$, we have*

$$x^2 exp(-\beta x^4) \leqslant sin(x)sinh(x) \leqslant x^2 exp\left(-\frac{x^4}{90}\right),$$

with $\beta = -ln(sin(\alpha)sinh(\alpha)/\alpha^2)/\alpha^4$.

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*Corresponding author.

Email address: `christophe.chesneau@unicaen.fr`* (Corresponding Author)

Proof. We have the following infinite products:

$$\frac{sin(x)}{x} = \prod_{k=1}^{+\infty}\left(1-\frac{x^2}{\pi^2k^2}\right), \quad \frac{sinh(x)}{x} = \prod_{k=1}^{+\infty}\left(1+\frac{x^2}{\pi^2k^2}\right).$$

Therefore, using the inequality $1+y \leqslant e^y$ for $y \in \mathbb{R}$ and $\zeta(4) = \sum_{k=1}^{+\infty}(1/k^4) = \pi^4/90$, we have

$$\begin{aligned} sin(x)sinh(x) &= x^2\prod_{k=1}^{+\infty}\left(1-\frac{x^2}{\pi^2k^2}\right)\prod_{k=1}^{+\infty}\left(1+\frac{x^2}{\pi^2k^2}\right) \\ &= x^2\prod_{k=1}^{+\infty}\left(1-\frac{x^4}{\pi^4k^4}\right) \leqslant x^2\prod_{k=1}^{+\infty} exp\left(-\frac{x^4}{\pi^4k^4}\right) \\ &= x^2 exp\left(-\frac{x^4}{\pi^4}\zeta(4)\right) = x^2 exp\left(-\frac{x^4}{90}\right). \end{aligned}$$

For the lower bound, using the Bernoulli inequality, i.e. for $u, v \in (0,1)$, we have $1-uv \geqslant (1-v)^u$, we obtain, for $x \in (0,\alpha)$,

$$\begin{aligned} sin(x)sinh(x) &= x^2\prod_{k=1}^{+\infty}\left(1-\frac{x^4}{\alpha^4}\frac{\alpha^4}{\pi^4k^4}\right) \geqslant x^2\prod_{k=1}^{+\infty}\left(1-\frac{\alpha^4}{\pi^4k^4}\right)^{x^4/\alpha^4} \\ &= x^2\left[\prod_{k=1}^{+\infty}\left(1-\frac{\alpha^4}{\pi^4k^4}\right)\right]^{x^4/\alpha^4} = x^2\left[\frac{sin(\alpha)sinh(\alpha)}{\alpha^2}\right]^{x^4/\alpha^4} \\ &= x^2 exp(-\beta x^4). \end{aligned}$$

This ends the proof of Proposition 2.1.

The proposition below proposes a tight polynomial lower bound for $sin(x)sinh(x)$.

Proposition 2.2. *For $x \in (0,\pi)$, we have*

$$x^2\left(1-\frac{x^4}{\pi^4}\right)^{\pi^4/90} \leqslant sin(x)sinh(x).$$

Proof. By using the Bernoulli inequality, i.e. for $u, v \in (0,1)$, we have $1-uv \geqslant (1-v)^u$, and $\zeta(4) = \pi^4/90$, we have

$$\begin{aligned} sin(x)sinh(x) &= x^2\prod_{k=1}^{+\infty}\left(1-\frac{x^4}{\pi^4k^4}\right) \geqslant x^2\prod_{k=1}^{+\infty}\left(1-\frac{x^4}{\pi^4}\right)^{1/k^4} \\ &= x^2\left(1-\frac{x^4}{\pi^4}\right)^{\zeta(4)} = x^2\left(1-\frac{x^4}{\pi^4}\right)^{\pi^4/90}. \end{aligned}$$

This ends the proof of Proposition 2.2.

Figure 1 illustrates the sharpness of the bounds in Propositions 2.1 and 2.2 for $x \in (0,3)$. We see that these inequalities are sharp, particularly for $x \in (0,1.5)$. At least for $x \in (1.5,1)$, the lower bound in Proposition 2.2 is clearly more sharp to the one in Proposition 2.1.

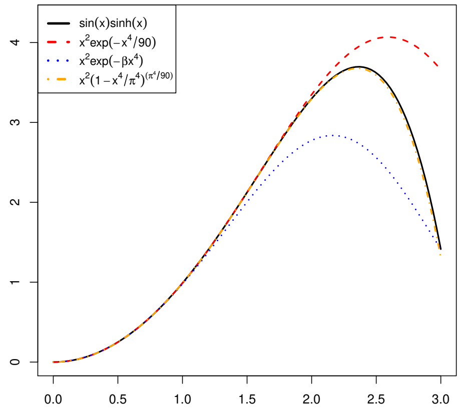

Figure 1: Graphs of the functions in Propositions 2.1 and 2.2 for $x \in (0, 3)$.

Similarly, the following result determines a double inequality for $cos(x)cosh(x)$, implying $cos(x)cosh(x) \leqslant 1$ for the upper bound.

Proposition 2.3. *For $x \in (0, \alpha)$ and $\alpha \in (0, \pi/2)$, we have*

$$exp(-\gamma x^4) \leqslant cos(x)cosh(x) \leqslant exp\left(-\frac{x^4}{6}\right),$$

with $\gamma = -ln(cos(\alpha)cosh(\alpha))/\alpha^4$.

Proof. We have the following infinite products:

$$cos(x) = \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right), \quad cosh(x) = \prod_{k=1}^{+\infty}\left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right).$$

Therefore, using the inequality $1+y \leqslant e^y$ for $y \in \mathbb{R}$ and $\sum_{k=1}^{+\infty}(1/(2k-1)^4) = (15/16)\zeta(4) = (15/16)(\pi^4/90) = (1/16)(\pi^4/6)$, we have

$$\begin{aligned} cos(x)cosh(x) &= \prod_{k=1}^{+\infty}\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right)\prod_{k=1}^{+\infty}\left(1 + \frac{4x^2}{\pi^2(2k-1)^2}\right) \\ &= \prod_{k=1}^{+\infty}\left(1 - \frac{16x^4}{\pi^4(2k-1)^4}\right) \leqslant \prod_{k=1}^{+\infty} exp\left(-\frac{16x^4}{\pi^4(2k-1)^4}\right) \\ &= exp\left(-x^4\frac{16}{\pi^4}\sum_{k=1}^{+\infty}\frac{1}{(2k-1)^4}\right) = exp\left(-\frac{x^4}{6}\right). \end{aligned}$$

By virtue of the Bernoulli inequality, i.e. for $u, v \in (0, 1)$, we have $1 - uv \geqslant (1 - v)^u$, we obtain, for $x \in (0, \alpha)$,

$$cos(x)cosh(x) = \prod_{k=1}^{+\infty}\left(1 - \frac{x^4}{\alpha^4}\frac{16\alpha^4}{\pi^4(2k-1)^4}\right) \geqslant \prod_{k=1}^{+\infty}\left(1 - \frac{16\alpha^4}{\pi^4(2k-1)^4}\right)^{x^4/\alpha^4}$$

$$= \left[\prod_{k=1}^{+\infty} \left(1 - \frac{16\alpha^4}{\pi^4 (2k-1)^4} \right) \right]^{x^4/\alpha^4} = [cos(\alpha)cosh(\alpha)]^{x^4/\alpha^4}$$
$$= exp(-\gamma x^4).$$

The proof of Proposition 2.3 is completed.

The proposition below presents a sharp polynomial lower bound for $cos(x)cosh(x)$.

Proposition 2.4. *For $x \in (0, \pi/2)$, we have*

$$\left(1 - \frac{16x^4}{\pi^4} \right)^{\pi^4/96} \leqslant cos(x)cosh(x).$$

Proof. By using the Bernoulli inequality, i.e. for $u, v \in (0,1)$, we have $1 - uv \geqslant (1-v)^u$, and $\sum_{k=1}^{+\infty} (1/(2k-1)^4) = \pi^4/96$, we have

$$cos(x)cosh(x) = \prod_{k=1}^{+\infty} \left(1 - \frac{16x^4}{\pi^4 (2k-1)^4} \right) \geqslant \prod_{k=1}^{+\infty} \left(1 - \frac{16x^4}{\pi^4} \right)^{1/(2k-1)^4}$$
$$= \left(1 - \frac{16x^4}{\pi^4} \right)^{\sum_{k=1}^{+\infty} (1/(2k-1)^4)} = \left(1 - \frac{16x^4}{\pi^4} \right)^{\pi^4/96}.$$

This ends the proof of Proposition 2.4.

Figure 2 illustrates the sharpness of the bounds in Propositions 2.3 and 2.4 for $x \in (0, 1.5)$. At least for $x \in (0.5, 1.5)$, we see that the lower bound in Proposition 2.4 is clearly more sharp to the one in Proposition 2.3.

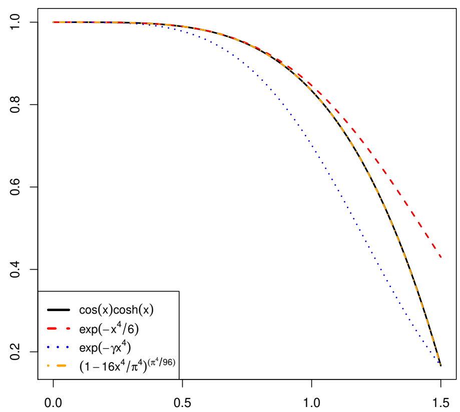

Figure 2: Graphs of the functions in Propositions 2.3 and 2.4 for $x \in (0, 1.5)$.

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