A note on finite union of primary submodules

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Abstract. Let M be a nonzero unital R-module. A proper submodule N of M is said to be a primary submodule if for a ∈ R, m ∈ M and whenever am ∈ N, then either a ∈ \sqrt{(N : M)} or m ∈ N. Atani and Tekir gave the Primary Avoidance Theorem as follows: let N be a submodule of M and N ⊆ \bigcup_{i=1}^{n} N_i be a covering of submodules of M, where at most two of N_i’s are not primary. If \sqrt{(N_i : M)} \notin \sqrt{(N_j : M)} for all i ≠ j, then N ⊆ N_k for some k ∈ {1, 2, ..., n} [1, Theorem 1]. In this paper, our aim is to improve the aforementioned version of Primary Avoidance Theorem and to obtain a similar result with weaker conditions.

2010 Mathematics Subject Classifications: 16P40, 13A15

Key Words and Phrases: prime submodules, primary submodules, Primary Avoidance Theorem

1. Introduction

In this paper, all rings under consideration will be assumed to be commutative with nonzero identity and all modules will be nonzero unital. Let R always denote such a ring and M denote such an R-module. Recall the following well-known theorem (Prime Avoidance Theorem) in commutative algebra: Suppose that I is an ideal of R and P_1, P_2, ..., P_n are prime ideals of R such that I ⊆ P_1 ∪ P_2 ∪ ... ∪ P_n. Then I ⊆ P_i for some i ∈ {1, 2, ..., n} [9]. Here, a question naturally arises: Is the aforementioned property of prime ideals always true if we replace the finite union by infinite one? This question has been studied in many papers. See, for example, [7], [8] and [4]. Let R be a ring. Then R is said to be compactly packed by primes or a compactly packed ring if an ideal I of R is contained in an arbitrary union of prime ideals, then I is contained in one of those primes [8]. In [7], the authors showed that a ring R is a compactly packed ring if and only if each prime ideal P of R is a radical of a principal ideal, namely, P = \sqrt{(a)} for some a ∈ R [7, Theorem 1].

As in the ring theory, the notion of prime submodule and its generalizations have a key role to determine the structure of a given module. Recall that a proper submodule N of M is said to be a prime submodule if for each a ∈ R, m ∈ M and am ∈ N, then either a ∈ (N : M) or m ∈ N, where (N : M) = \{r ∈ R : rM ⊆ N\} denotes the residual of N by M. In this case, (N : M) = p is a prime ideal of R and N is said to be p-prime [5]. C. P. Lu, in his paper [5], transferred the Prime Avoidance Theorem to modules as follows: if N ⊆ N_1 ∪ N_2 ∪ ... ∪ N_n, where N is a submodule of M and at most two of N_i’s are not prime submodules, under the condition

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that \((N_i : M) \not\supseteq (N_j : M)\) for all \(i \neq j\), then \(N \subseteq N_k\) for some \(k \in \{1, 2, \ldots, n\}\) [5, Theorem 2.3]. Afterwards, Jafari showed that the Prime Avodiance Theorem is still true for modules replacing the condition \((N_i : M) \not\supseteq (N_j : M)\) by a weaker one \((N_i : M) \not= (N_j : M)\). He proved that if \(N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n\), where \(N_i\) is a \(p_i\)-prime submodule, \(N\) is a submodule and all \(p_i\)'s are distinct, then \(N \subseteq N_k\) for some \(k \in \{1, 2, \ldots, n\}\) [2, Proposition 2.1]. The authors, in [1], gave the Primary Avodiance Theorem which is a generalization of the Prime Avodiance Theorem in [5]. Recall from [1], a proper submodule \(N\) of \(M\) is said to be a primary submodule if for \(a \in R, m \in M\) and \(am \in N\) imply either \(a^n \in (N : M)\) or \(m \in N\) for some \(n \in N\). In this case, the radical \(\sqrt{(N : M)} = p\) of \((N : M)\) is a prime ideal and \(N\) is said to be \(p\)-primary. Note that every prime submodule is primary and the converse is true in case \(\sqrt{(N : M)} = (N : M)\). In [1], the authors proved that if \(N\) is a submodule of \(M\) and \(N \subseteq \bigcup_{i=1}^{n} N_i\), where at most two of \(N_i\)'s are not primary submodules and \(\sqrt{(N_i : M)} \not\subseteq \sqrt{(N_j : M)}\) for all \(i \neq j\), then \(N \subseteq N_k\) for some \(k \in \{1, 2, \ldots, n\}\) [1, Theorem 1]. Our aim in this article is to improve the result [1, Theorem 1] putting by weaker conditions on primary submodules in the covering (See, Theorem 2 and Theorem 5).

2. Primary Avoidance Theorem

In this section, our aim is to prove a new version of Primary Avoidance Theorem for modules. Now, we begin with the following lemma which will be used later.

**Lemma 1.** Let \(M\) be an \(R\)-module and \(N_i\) be a \(p_i\)-primary submodule for each \(i = 1, 2, \ldots, n\). If \(M = \bigcup_{i=1}^{n} N_i\) and all \(p_i\)'s are distinct primes, then \(M = N_k\) for some \(k \in \{1, 2, \ldots, n\}\).

**Proof.** We use induction on \(n\). If \(n = 1\) or 2, then the claim is clear. Assume that the claim is true for each \(k \leq n\). Now, let \(M = \bigcup_{i=1}^{n+1} N_i\) for some \(p_i\)-primary submodules \(N_i\) of \(M\), where all \(p_i\)'s are distinct. Consider the set \(\Omega = \{p_1, p_2, \ldots, p_{n+1}\}\). Then \(\Omega\) has a minimal element with respect to inclusion "\(\subseteq\)". Assume that \(p_{n+1} \in \Omega\) is the minimal element so that \(p_k \not\subseteq p_{n+1}\) for each \(k \in \{1, 2, \ldots, n\}\). Now, we will show that \(N_{n+1} \subseteq \bigcup_{i=1}^{n} N_i\). Suppose to the contrary. Then \(N_{n+1} \not\subseteq \bigcup_{i=1}^{n} N_i\). Now, put \(K = \bigcap_{i=1}^{n+1} N_i\). Then \(M/K = \bigcup_{i=1}^{n+1} (N_i/K)\), \(\bigcap_{i=1}^{n+1} (N_i/K) = 0_{M/K}\) and all \(N_i/K\) are \(p_i\)-primary submodules of \(M/K\). Thus, without loss of generality, we may assume that \(\bigcap_{i=1}^{n+1} N_i = 0\). Now, consider the homomorphism \(\pi : M \to \bigoplus_{i=1}^{n+1} (M/N_i)\) defined by \(\pi(m) = (m + N_1, m + N_2, \ldots, m + N_{n+1})\) for each \(m \in M\). As \(\bigcap_{i=1}^{n+1} N_i = 0\), \(f\) is monomorphism so that \(M\) can be considered as a submodule of \(\bigoplus_{i=1}^{n+1} (M/N_i)\). This implies that \(\bigcap_{i=1}^{n+1} p_i = \bigcap_{i=1}^{n+1} (N_i : M) = ann_R(M) \subseteq \bigcap_{i=1}^{n+1} (N_i : M) = \bigcap_{i=1}^{n+1} p_i\) and so \(ann_R(M) = \bigcap_{i=1}^{n+1} p_i\). Also, if \(\bigcap_{i=1}^{n} N_i = 0\), then similarly \(ann_R(M) = \bigcap_{i=1}^{n} p_i = ann_R(M)\). This yields that
ann\(_R(\oplus_{i=1}^n (M/N_i)) = \bigcap_{i=1}^n p_i = ann\(_R(M) = \bigcap_{i=1}^{n+1} p_i \subseteq p_{n+1}. As p_{n+1} is a prime ideal, we conclude that p_k \subseteq p_{n+1} for some 1 \leq k \leq n, a contradiction. Thus \bigcap_{i=1}^n N_i \neq 0 and so there exists 0 \neq m \in \bigcap_{i=1}^n N_i. As \bigcap_{i=1}^{n+1} N_i = 0, note that m \notin N_{n+1}. Now, we will show that N_{n+1} \subseteq \bigcup_{i=1}^n N_i. To see this, take an element m' \in N_{n+1}. Since m \notin N_{n+1}, we have m + m' \notin N_{n+1}. As m + m' \in M = \bigcup_{i=1}^n N_i, there exists k \in \{1,2,\ldots,n\} such that m + m' \in N_k. Since m \in N_k, we get m = (m + m') - m \in N_k \subseteq \bigcap_{i=1}^n N_i and so N_{n+1} \subseteq \bigcup_{i=1}^n N_i. Then we have M = \bigcup_{i=1}^n N_i, by induction hypothesis, we get M = N_k for some k \in \{1,2,\ldots,n\}.

Now, we are ready to prove Primary Avoidance Theorem.

**Theorem 2.** (Primary Avoidance Theorem) Let M be an R-module, N a submodule and N\(_i\) be p\(_i\)-primary submodule for each i = 1,2,\ldots,n. If N \subseteq \bigcup_{i=1}^n N_i and all p\(_i\)'s are distinct primes, then N \subseteq N_k for some k \in \{1,2,\ldots,n\}.

**Proof.** We use induction on n. The cases n = 1 and 2 are trivial. So assume that the claim is true for all k \leq n. Suppose that N \subseteq \bigcup_{i=1}^{n+1} N_i, where N\(_i\) is p\(_i\)-primary and all p\(_i\)'s are distinct. Now, we will show that N \subseteq N_k for some k \in \{1,2,\ldots,n+1\}. Suppose to the contrary, that is, N \nsubseteq N_k for all k \in \{1,2,\ldots,n+1\}. By the covering N \subseteq \bigcup_{i=1}^{n+1} N_i, we have N = \bigcup_{i=1}^{n+1} (N \cap N_i). It is clear that (N \cap N_i : N) = (N_i : N) and also p\(_i\) = (N\(_i\) : M) \subseteq (N\(_i\) : N). Let r \in (N\(_i\) : N). Then rN \subseteq N\(_i\). As N\(_i\) is a primary submodule and N \nsubseteq N\(_i\), we can easily get r \in \sqrt{(N\(_i\) : M)} = p\(_i\). Thus we have p\(_i\) = (N \cap N\(_i\) : N). Also note that N \cap N\(_i\) is a primary submodule of N since N\(_i\) is a primary submodule of M with N \nsubseteq N\(_i\). As N = \bigcup_{i=1}^{n+1} (N \cap N\(_i\)), by Lemma 1, we have N = N \cap N\(_k\) \subseteq N\(_k\) for some k \in \{1,2,\ldots,n+1\}, a contradiction. Thus, we have N \subseteq N_k for some k \in \{1,2,\ldots,n+1\}.

**Theorem 3.** (2, Proposition 2.1) Let M be an R-module and N \subseteq \bigcup_{i=1}^n N_i, where N is a submodule of M and N\(_i\) is a p\(_i\)-prime submodule of M. If all p\(_i\)'s are distinct primes, then N \subseteq N_k for some k \in \{1,2,\ldots,n\}.

**Proof.** Follows from Theorem 2.

The condition "all p\(_i\)'s are distinct" in Theorem 2 is essential. See the following example.

**Example 4.** Let F = \(\mathbb{Z}_2\) and M = \(F^3\). Then M is a vector space over the field F. So all proper subspace of M is \{0\}-primary submodule. Consider the basis \{e\(_1\), e\(_2\), e\(_3\)\} of M. And put

\[
N_1 = Fe_1 + Fe_2, \quad N_2 = Fe_1 + Fe_3, \\
N_3 = F(e_2 + e_3), \quad N = Fe_2 + Fe_3.
\]

Then note that N \subseteq \bigcup_{i=1}^3 N_i but N \nsubseteq N\(_i\) for all i = 1,2,3.
Recall that for a submodule $N$ of $M$, the prime radical of $N$, denoted by $\text{rad}(N)$, is defined to be the intersection of all prime submodules containing $N$. If there is no such prime submodule, we say $\text{rad}(N) = M$. Also, recall that an $R$-module $M$ is said to be a multiplication module if and only if $\sqrt{(N : M)} = (\text{rad}(N) : M)$ [6, Theorem 4.4].

**Theorem 5.** Let $M$ be a finitely generated multiplication module and $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$, where $N_i$’s are $p_i$-primary submodule and $N$ is a submodule of $M$. If $\text{rad}(N_i) \neq \text{rad}(N_j)$ for each $i \neq j$, then $N \subseteq N_k$ for some $k \in \{1, 2, \ldots, n\}$.

**Proof.** Now, we will show that all $p_i$’s are distinct. Suppose that $p_i = p_j$ for some $i \neq j$. As $M$ is finitely generated, by [6, Theorem 4.4], $p_i = \sqrt{(N_i : M)} = (\text{rad}(N_i) : M)$. This implies that $(\text{rad}(N_i) : M) = (\text{rad}(N_j) : M)$ since $p_i = p_j$. As $M$ is multiplication module, $\text{rad}(N_i) = (\text{rad}(N_i) : M)M = (\text{rad}(N_j) : M)M = \text{rad}(N_j)$, a contradiction. Thus all $p_i$’s are distinct. Then by Theorem 2, $N \subseteq N_k$ for some $k \in \{1, 2, \ldots, n\}$.

**Acknowledgement.** We would like to thank the referee for her/his all efforts in proofreading the manuscript.

**References**


