EURASIAN BULLETIN OF MATHEMATICS
EBM (2021), VOL. 4, NO. 2, 71-97

The Marshall-Olkin-Odd Weibull-G Family of Distributions: Model, Properties and Applications

Boikanyo Makubate[1], Fastel Chipepa[1,*], Broderick Oluyede[1], Simbarashe Chamunorwa[1]

Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Palapye, Bostwana.[1]

Abstract. A new generalization of the odd Weibull-G distribution is developed to produce a new family of distributions, namely, the Marshall-Olkin Odd-Weibull (MO-OW-G) family of distributions. The new family of distributions is based on the generator derived by Marshall and Olkin [27]. The statistical properties of the new proposed distribution are studied. We also present maximum likelihood estimators for the model parameters. We then demonstrate the performance of the new model through conducting a simulation study. Finally, real life data examples are given to demonstrate the usefulness and applicability of the proposed model.
2010 Mathematics Subject Classifications: 62E05
Key Words and Phrases: Marshall-Olkin-G, Odd-Weibull-G, Maximum Likelihood Estimation

---

1. Introduction

There is an increase in the demand for distributions which can handle various levels of skewness, kurtosis and which can model data that have both monotonic and non-monotonic hazard rate functions. With this increased demand, a lot of important new generalized distributions have been proposed in the literature. In this paper, we present and study the mathematical properties of the Marshall-Olkin Odd Weibull-G family of distributions. This class of distributions is flexible in accommodating all forms of hazard rate functions and modelling complex data relating to reliability studies and other lifetime data.

Development of new extended and generalized distributions for modelling monotone and non-monotone hazard rates functions, particularly in reliability studies, statistical mechanics, quality control, economics are very crucial. Generalized Weibull-type distributions and related distributions including modified Weibull distributions (see Oluyede et al. [30], Pham et al. [31] and references therein for more details) has produced distributions that can handle heavy tailed data and play crucial role in the modelling of monotone and non-monotone hazard rate functions.

Extensions of the Marshall-Olkin distribution include the Kumaraswamy Marshall-Olkin-G (KwMO-G) family [4], beta Marshall-Olkin-G (BMO-G) family [2], Marshall-Olkin-G (MO-G) family [27], Topp-Leone-Marshall-Olkin-G (TL-MO-G) family [15], Marshall-Olkin Log-logistic Extended

---

*Corresponding author.

Email address: `chipepa.fastel@studentmail.biust.ac.bw* (Corresponding author)`

Weibull (MOLLEW) family [26], among others. These generalized distributions provided extra flexibility in data modeling in a wide range of fields including science, engineering, finance and medicine.

Additional well-known generalized distributions include the beta odd log-logistic generalized-G by [20], the transmuted-G by [34], odd log-logistic-G by [23], the gamma-G by [36], the Kumaraswamy-G by [18], the logistic-G by [35], exponentiated generalized-G by [19], beta-G by [22], T-X family by [6], generalized transmuted-G by [29], generalized odd log-logistic-G by [21]. Some structural properties of the generalized distributions may be easily obtained using mixture forms of exponentiated-G (exp-G) distributions. It is in the same vein that we develop the new family of distributions called the Marshall-Olkin Odd Weibull-G (MO-OW-G) family of distributions.

Marshall and Olkin [27], introduced a new distribution with cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x;\delta,\psi) = 1 - \frac{\delta\bar{G}(x;\psi)}{1-\bar{\delta}\bar{G}(x;\psi)}, \tag{1.1}$$

and

$$f_{MO-G}(x;\delta,\psi) = \frac{\delta g(x;\psi)}{\left[1-\bar{\delta}\bar{G}(x;\psi)\right]^2}, \tag{1.2}$$

respectively, where δ is the tilt parameter, $\overline{\delta} = 1-\delta$, $G(x;\psi)$ is the baseline cdf, $\overline{G}(x;\psi) = 1 - G(x;\psi)$ and ψ is the vector of parameters from the baseline distribution. The distribution is more flexible compared to the exponential, Weibull and gamma distributions.

Furthermore, Bourguignon et al. [11], developed the odd Weibull-G family of distributions with the cdf and pdf given by

$$F_{OW-G}(x;\lambda,\beta,\psi) = 1 - \exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right), \tag{1.3}$$

$$f_{OW-G}(x;\lambda,\beta,\psi) = \frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right), \tag{1.4}$$

respectively, where $G(x;\psi)$ is the cdf of the baseline distribution $\beta > 0$ is a shape parameter, and $\lambda > 0$ is a scale parameter.

In this paper, we develop a new family of distributions, namely, the MO-OW-G family of distributions. In Section 2, we present the new generalized family of distributions and the linear representation. Statistical and mathematical properties including the distribution of order statistics, Rényi entropy, moments, generating and quantile functions are presented in Section 3. In Section 4, we present the maximum likelihood estimates. Some sub-families of the MO-OW-G family of distributions are presented in Section 5. Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter in Section 6. Applications of the proposed model to real data are given in Section 7, followed by concluding remarks.

2. Marshall-Olkin odd Weibull-G Family of Distributions

We derive a new class of distributions, namely, the MO-OW-G distribution. The new distribution is developed using the generalization proposed by Marshall and Olkin [27] and taking the baseline distribution to be the OW-G distribution by Bourguignon et al. [11]. The cdf and pdf of the MO-OW-G family of distributions are given by

$$F_{MO-OW-G}(x;\beta,\delta,\lambda,\psi) = 1 - \left(\frac{\delta \exp\left(-\lambda \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)}{1-\overline{\delta}\exp\left(-\lambda \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)} \right) \tag{2.1}$$

and

$$f_{MO-OW-G}(x;\beta,\delta,\lambda,\psi) = \frac{\lambda\beta\delta g(x;\psi)G(x;\psi)^{\beta-1}\exp\left(-\lambda \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)}{\overline{G}(x;\psi)^{\beta+1}\left(1-\overline{\delta}\exp\left(-\lambda \left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right)^{2}}, \tag{2.2}$$

respectively, for $\delta > 0$, $\bar{\delta} = 1-\delta$ and ψ is a vector of parameters from the baseline distribution function G(.).

2.1. Linear Representation

In this section, we present the series expansion of the MO-OW-G family of distributions using general results for the Marshall and Olkin's family of distributions as presented by Barreto-Souza et al. [8]. Considering

$$f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) = \frac{\delta f_{OW-G}(x;\lambda,\beta,\psi)}{(1-\bar{\delta}\bar{F}_{OW-G}(x;\lambda,\beta,\psi))^2}, \tag{2.3}$$

we can write equation (2.2) as

$$f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) = \frac{f_{OW-G}(x;\lambda,\beta,\psi)}{\delta[1-\frac{\delta-1}{\delta}F_{OW-G}(x;\lambda,\beta,\psi)]^2}, \tag{2.4}$$

where $f_{OW-G}(x;\lambda,\beta,\psi)$ and $F_{OW-G}(x;\lambda,\beta,\psi)$ are as given in equations (1.4) and (1.3), respectively. Also we apply the series expansion

$$(1-l)^{-n} = \sum_{j=0}^{\infty}\frac{\Gamma(n+j)}{\Gamma(n)j!}l^j, \tag{2.5}$$

which is valid for $|l| < 1$ and $n > 0$. If $\delta \in (0,1)$, we can obtain

$$f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) = f_{OW-G}(x;\lambda,\beta,\psi)\sum_{j=0}^{\infty}\sum_{e=0}^{j} w_{j,e}F_{OW-G}(x;\lambda,\beta,\psi)^{j-e}, \tag{2.6}$$

where $w_{j,e} = w_{j,e}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-e}\binom{j}{e}$. For $\delta > 1$, we have

$$f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) = f_{OW-G}(x;\lambda,\beta,\psi)\sum_{j=0}^{\infty} v_j F^j_{OW-G}(x;\lambda,\beta,\psi), \tag{2.7}$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-1/\delta)}{\delta}$.

Therefore, for $\delta \in (0,1)$, equation (2.2) becomes

$$\begin{aligned} f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) &= \frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\ &\times \sum_{j=0}^{\infty}\sum_{k=0}^{j} w_{j,e}\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j-e}\\ &= \sum_{j=0}^{\infty}\sum_{e=0}^{j}\frac{\beta\lambda w_{j,e} g(x)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\\ &\times \exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j-e}. \end{aligned}$$

Applying the following series expansions

$$\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j-e} = \sum_{q=0}^{\infty}(-1)^q\binom{j-e}{q}\exp\left(-\lambda q\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\Bigg],$$

$$\exp\left(-\lambda(q+1)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) = \sum_{z=0}^{\infty}\frac{(-1)^z(-\lambda(q+1))^z}{z!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta z}$$

and

$$\overline{G}^{-(\beta(z+1)+1)}(x;\psi) = \sum_{m=0}^{\infty}(-1)^m\binom{-(\beta(z+1)+1)}{m}G^m(x;\psi),$$

we can write

$$\begin{aligned} f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) &= \sum_{j,q,z,m=0}^{\infty}\sum_{e=0}^{j}\frac{(-1)^{q+z+m}(-\lambda(q+1))^z\beta\lambda}{(\beta(z+1)+m)z!}w_{j,k}\\ &\times \binom{j-e}{q}\binom{-(\beta(z+1)+1)}{m}(\beta(z+1)+m)\\ &\times g(x;\psi)[G(x;\psi)]^{\beta(z+1)+m-1}\\ &= \sum_{j,q,z,m=0}^{\infty} w^*_{j,q,z,m}\, g_{\beta(z+1)+m}(x;\psi). \end{aligned} \tag{2.8}$$

It follows that for $\delta \in (0,1)$, the MO-OW-G family of distributions can be expressed as a linear combination of the exponentiated-G (Exp-G) distribution with power parameter $(\beta(z+1)+m)$ and linear component

$$\begin{aligned} w^*_{j,q,z,m} &= \sum_{e=0}^{j}\frac{(-1)^{q+z+m}(-\lambda(q+1))^z\beta\lambda}{(\beta(z+1)+m)z!}w_{j,k}\\ &\times \binom{j-e}{q}\binom{-(\beta(z+1)+1)}{m}. \end{aligned} \tag{2.9}$$

Furthermore, for $\delta > 1$ equation (2.2) can be written as

$$\begin{aligned} f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) &= \frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\ &\times \sum_{j=0}^{\infty} v_j\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^j\\ &= \sum_{j=0}^{\infty} v_j\frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\ &\times \left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^j. \end{aligned}$$

Applying the series expansions

$$\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^j = \sum_{p=0}^{\infty}(-1)^p\binom{j}{p}\exp\left(-\lambda p\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\Bigg],$$

$$\exp\left(-\lambda(p+1)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) = \sum_{w=0}^{\infty}\frac{(-1)^w(-\lambda(p+1))^w}{w!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta w}$$

and

$$\overline{G}^{-(\beta(w+1)+1)}(x;\psi) = \sum_{b=0}^{\infty}(-1)^b\binom{-(\beta(w+1)+1)}{b}G^b(x;\psi),$$

we can write

$$\begin{aligned} f_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) &= \sum_{j,p,w,b=0}^{\infty}\frac{(-1)^{p+w+b}(-\lambda(p+1))^w\beta\lambda}{(\beta(w+1)+b)w!}v_j\\ &\times \binom{j}{p}\binom{-(\beta(w+1)+1)}{b}(\beta(w+1)+b)\\ &\times g(x;\psi)[G(x;\psi)]^{\beta(w+1)+b-1}\\ &= \sum_{j,p,w,b=0}^{\infty} v^*_{j,p,w,b}\, g_{\beta(w+1)+b}(x;\psi). \end{aligned} \quad (2.10)$$

Therefore, for $\delta > 1$, the MO-OW-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(\beta(w+1)+b)$ and linear component

$$\begin{aligned} v^*_{j,p,w,b} &= \frac{(-1)^{p+w+b}(-\lambda(p+1))^w\beta\lambda}{(\beta(w+1)+b)w!}v_j\\ &\times \binom{j}{p}\binom{-(\beta(w+1)+1)}{b}. \end{aligned} \quad (2.11)$$

2.2. Quantile Function

The quantile function for the MO-OW-G family of distributions is obtained by solving the following equation:

Table 1: Table of Quantiles for Selected Parameters of the MO-OW-LLoG Distribution

u	(1.5,1.5,1.5,1.5)	(1.5,1,1.5,1.5)	(1.5,0.5,1.5,1)	(1.5,1.5,1,1.5)	(1,1.5,1,0.5)
0.1	0.3638	0.4356	0.4564	0.3072	0.0049
0.2	0.5022	0.6014	0.7402	0.4287	0.0221
0.3	0.6117	0.7325	0.9952	0.5281	0.0565
0.4	0.7096	0.8497	1.2433	0.6196	0.1160
0.5	0.8033	0.9619	1.4975	0.7096	0.2135
0.6	0.8984	1.0758	1.7712	0.8032	0.3732
0.7	1.0012	1.1989	2.0839	0.9069	0.6442
0.8	1.1226	1.3443	2.4742	1.0318	1.1512
0.9	1.2930	1.5483	3.0583	1.2098	2.3564

$$1-\left(\frac{\delta\exp\left[-\lambda\left(\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right)^{\beta}\right]}{1-\overline{\delta}\exp\left[-\lambda\left(\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right)^{\beta}\right]}\right)=u$$

for $0\leq u\leq 1$, that is,

$$\frac{1-u}{\delta+\overline{\delta}(1-u)}=\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right).$$

Therefore, the quantiles of the MO-OW-G family of distributions may be determined by solving the non-linear equation

$$x(u)=G^{-1}\left(\frac{\left[-\frac{1}{\lambda}\log\left(\frac{1-u}{\delta+\overline{\delta}(1-u)}\right)\right]^{\frac{1}{\beta}}}{1+\left[-\frac{1}{\lambda}\log\left(\frac{1-u}{\delta+\overline{\delta}(1-u)}\right)\right]^{\frac{1}{\beta}}}\right), \tag{2.12}$$

via iterative methods in R or Matlab software. Quantiles for selected parameter values for the Marshall-Olkin odd Weibull-log-logistic (MO-OW-LLoG) distribution are shown in Table 1.

3. Some Statistical Properties

We present some statistical properties of the MO-OW-G family of distributions, which include order statistics, entropy, moments, incomplete moments, generating function and probability weighted moments (PWMs).

3.1. Distribution of Order Statistics

Suppose that $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d) random variables distributed according to (2.2). The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x;\delta,\lambda,\beta\psi)=\delta n! f_{OW-G}(x;\lambda,\beta,\psi)\sum_{l=0}^{n-i}\frac{(-1)^l}{(i-1)!(n-i)!}\frac{F_{OW-G}^{l+i-1}(x;\lambda,\beta,\psi)}{[1-\overline{\delta}F_{OW-G}(x;\lambda,\beta,\psi)]^{l+i-1}}. \tag{3.1}$$

If $\delta \in (0,1)$, we have

$$f_{i:n}(x;\delta,\lambda,\beta,\psi) = f_{_{OW-G}}(x;\lambda,\beta,\psi)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j} U_{j,l,k} F_{_{OW-G}}^{j+l-k+i-1}(x;\lambda,\beta,\psi), \tag{3.2}$$

where

$$U_{j,l,k} = U_{j,l,k}(\delta) = \frac{\delta n!(-1)^l(1-\delta)^j(-1)^{j-k}}{(i-1)!(n-i)!}\binom{j}{k}\binom{l+i+j}{j}. \tag{3.3}$$

For $\delta > 1$, we write $1-\bar{\delta}\bar{F}_{_{OW-G}}(x;\lambda,\beta,\psi) = \delta\left\{1-(\delta-1)F_{_{OW-G}}(x;\lambda,\beta,\psi)/\delta\right\}$, such that

$$f_{i:n}(x;\delta,\lambda,\beta,\psi) = f_{_{OW-G}}(x;\lambda,\beta,\psi)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i} C_{j,l} F_{_{OW-G}}^{j+l+i-1}(x;\lambda,\beta,\psi), \tag{3.4}$$

where

$$C_{j,l} = C_{j,l}(\delta) = \frac{(-1)^l(\delta-1)^j n!}{\delta^{l+j+i}(i-1)!(n-i)!}\binom{l+i+j}{j}. \tag{3.5}$$

For $\delta \in (0,1)$, using equation (3.2) and substituting $f(x)$ by equation (1.4) and $F_{OW-G}(x;\lambda,\beta,\psi)$ by equation (1.3), we get

$$\begin{aligned}
f_{i:n}(x;\beta,\lambda,\delta,\psi) &= \frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\
&\times \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j} U_{j,l,k}\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l-k+i-1}\\
&= \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j} U_{j,l,k}\frac{\lambda\beta g(x;\psi)}{\overline{G}^2(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\
&\times \left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l-k+i-1}.
\end{aligned}$$

By applying the expansions

$$\begin{aligned}
\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l-k+i-1} &= \sum_{m=0}^{\infty}(-1)^m\binom{j+l-k+i+1}{m}\\
&\times \exp\left(-\lambda m\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right),
\end{aligned}$$

$$\exp\left(-\lambda(m+1)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) = \sum_{w=0}^{\infty}\frac{(-1)^w(-\lambda(m+1))^w}{w!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta w}$$

and

$$\overline{G}^{-(\beta(w+1)+1)}(x;\psi) = \sum_{p=0}^{\infty}\binom{-(\beta(w+1)+1)}{p}G^p(x;\psi),$$

we can write

$$f_{i:n}(x;\beta,\lambda,\delta,\psi) = \sum_{j,m,w,p=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\frac{(-1)^{m+w}(-\lambda(m+1))^w}{(\beta(w+1)+p)w!}U_{j,l,k}\binom{(j+l-k+i+1)}{m}$$

$$\begin{aligned}
&\times \binom{-(\beta(w+1)+1)}{p}(\beta(w+1)+p)g(x;\psi)[G(x;\psi)]^{\beta(w+1)+p-1}\\
&= \sum_{j,m,w,p=0}^{\infty} U^{*}_{j,m,w,p}\, g_{\beta(w+1)+p}(x;\xi), \qquad (3.6)
\end{aligned}$$

where $g_{\beta(w+1)+p}(x;\psi) = (\beta(w+1)+p)g(x;\psi)[G(x;\psi)]^{\beta(w+1)+p-1}$ is an Exp-G distribution with power parameter $(\beta(w+1)+p)$ and

$$\begin{aligned}
U^{*}_{j,m,w,p} &= \sum_{l=0}^{n-i}\sum_{k=0}^{j} \frac{(-1)^{m+w}(-\lambda(m+1))^{w}}{(\beta(w+1)+p)w!} U_{j,l,k}\binom{(j+l-k+i+1)}{m}\\
&\times \binom{-(\beta(w+1)+1)}{p}. \qquad (3.7)
\end{aligned}$$

Furthermore, for $\delta > 1$, we get

$$\begin{aligned}
f_{i:n}(x;\beta,\lambda,\delta,\psi) &= \frac{\lambda\beta g(x;\psi)}{\overline{G}^{2}(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\sum_{j=0}^{\infty}\sum_{i=0}^{n-i} C_{j,l}\\
&\times \left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l+i-1}\\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{n-i} C_{j,l}\frac{\lambda\beta g(x;\psi)}{\overline{G}^{2}(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta-1}\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\\
&\times \left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l+i-1}.
\end{aligned}$$

By applying the series expansions

$$\begin{aligned}
\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j+l+i-1} &= \sum_{z=o}^{\infty}(-1)^{z}\binom{j+l+i+1}{z}\\
&\times \exp\left(-\lambda z\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right),
\end{aligned}$$

$$\exp\left(-\lambda(z+1)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) = \sum_{b=0}^{\infty}\frac{(-1)^{b}(-\lambda(z+1))^{b}}{b!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta b}$$

and

$$\overline{G}^{\beta(b+1)+1}(x;\psi) = \sum_{w=0}^{\infty}(-1)^{w}\binom{-(\beta(b+1)+1)}{w}G^{w}(x;\psi),$$

we can write

$$\begin{aligned}
f_{i:n}(x;\beta,\lambda,\delta,\psi) &= \sum_{j,z,b,w=0}^{\infty}\sum_{i=0}^{n-i}\frac{(-1)^{z+b+w}(-\lambda(z+1))^{b}}{\beta(b+1)+w} C_{j,l}\binom{j+l+i+1}{z}\\
&\times \binom{-(\beta(b+1)+1)}{b}(\beta(b+1)+w)g(x;\psi)[G(x;\psi)]^{\beta(b+1)+w-1}
\end{aligned}$$

$$= \sum_{j,z,b,w=0}^{\infty} C^{*}_{j,z,b,w} g_{\beta(b+1)+w}(x;\psi), \tag{3.8}$$

where $g_{\beta(b+1)+w}(x;\psi) = (\beta(b+1)+w)g(x;\psi)[G(x;\psi)]^{\beta(b+1)+w-1}$ is an Exp-G distribution with power parameter $(\beta(b+1)+w)$ and

$$\begin{aligned} C^{*}_{j,z,b,w} &= \sum_{i=0}^{n-i} \frac{(-1)^{z+b+w}(-\lambda(z+1))^{b}}{\beta(b+1)+w} C_{j,l} \binom{j+l+i+1}{z} \\ &\times \binom{-(\beta(b+1)+1)}{b}. \end{aligned} \tag{3.9}$$

3.2. Entropy

An Entropy is a measure of variation of uncertainty for a random variable X with pdf $f(x)$. There are several measure of entropy in the literature including Shannon entropy [33] and Rényi entropy [32]. Rényi entropy is defined by

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\int_0^{\infty} f^{\nu}(x)dx\right],$$

where $\nu > 0$ and $\nu \neq 1$. Using expansion (2.5), for $\delta \in (0,1)$

$$\begin{aligned} f^{\nu}_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) &= \frac{\delta^{\nu} f^{\nu}_{OW-G}(x;\lambda,\beta,\psi)}{\Gamma(2\nu)} \sum_{j=0}^{\infty}(1-\alpha)^{j}\Gamma(2\nu+j) \\ &\times \frac{[1-F_{OW-G}(x;\lambda,\beta,\psi)]^{j}}{j!} \end{aligned}$$

and for $\delta > 1$

$$f^{\nu}_{MO-OW-G}(x;\beta,\lambda,\delta,\psi) = \frac{f^{\nu}_{OW-G}(x)}{\delta^{\nu}\Gamma(2\nu)} \sum_{j=0}^{\infty}(\delta-1)^{j}\Gamma(2\nu+j)\frac{F^{j}_{OW-G}(x;\beta,\lambda,\psi)}{j!}.$$

Thus, Rényi entropy for $\delta \in (0,1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty} e_j \int_0^{\infty} f^{\nu}_{OW-G}(x;\lambda,\beta,\psi)(1-F_{OW-G}(x;\lambda,\beta,\psi))^{j}dx\right) \tag{3.10}$$

and

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f^{\nu}_{OW-G}(x;\lambda,\beta,\psi)F^{j}_{OW-G}(x;\lambda,\beta,\psi)dx\right), \tag{3.11}$$

respectively, where

$$e_j = e_j(\delta) = \frac{\delta^{\nu}(1-\delta)^{j}\Gamma(2\nu+j)}{\Gamma(2\nu)j!} \tag{3.12}$$

and

$$h_j = h_j(\delta) = \frac{(\delta-1)^{j}\Gamma(2\nu+j)}{\delta^{\nu+j}\Gamma(2\nu)j!}. \tag{3.13}$$

Now, for $\delta \in (0,1)$ and using equation (3.10), we have

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j=0}^{\infty} e_j \int_0^{\infty} \frac{\lambda^{\nu}\beta^{\nu}g^{\nu}(x;\psi)}{\overline{G}^{2\nu}(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\nu(\beta-1)} \\ &\times \exp\left(-\lambda\nu\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\left[1-\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]\right]^{j} dx\Bigg] \\ &= (1-\nu)^{-1}\log\Bigg[\sum_{j=0}^{\infty} e_j \int_0^{\infty} \frac{\lambda^{\nu}\beta^{\nu}g^{\nu}(x;\psi)}{\overline{G}^{2\nu}(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\nu\beta-1} \\ &\times \exp\left(-\lambda(v+j)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) dx\Bigg]. \end{aligned}$$

By considering the following expansions

$$\exp\left(-\lambda(\nu+j)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right) = \sum_{w=0}^{\infty}\frac{(-1)^{w}(-\lambda(\nu+j))^{w}}{w!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta w},$$

and

$$\overline{G}^{-(\beta(\nu+w)+\nu)}(x;\psi) = \sum_{m=0}^{\infty}\binom{-(\beta(\nu+w)+\nu)}{m}G^{m}(x;\psi),$$

we can write

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j,w,m=0}^{\infty} e_j\frac{\beta^{\nu}\lambda^{\nu}(-1)^{w}(-\lambda(\nu+j))^{w}}{w!} \\ &\times \binom{-(\beta(\nu+w)+\nu)}{m}\frac{1}{\left(\frac{\beta(\nu+w)-\nu+m}{\nu}+1\right)^{\nu}} \\ &\times \int_0^{\infty}\left(\left(\frac{\beta(\nu+w)-\nu+m}{\nu}+1\right)g(x;\psi)[G(x;\psi)]^{\frac{\beta(\nu+w)-\nu+m}{\nu}}\right)^{\nu} dx\Bigg] \\ &= (1-\nu)^{-1}\log\Bigg[\sum_{j,w,m=0}^{\infty} e^{*}_{j,w,m}\exp(1-\nu)I_{REG}\Bigg], \end{aligned} \tag{3.14}$$

where

$$e^{*}_{j,w,m} = e_j\frac{\beta^{\nu}\lambda^{\nu}(-1)^{w+m}(-\lambda(\nu+j))^{w}}{w!}\binom{-(\beta(\nu+w)+\nu)}{m}\frac{1}{\left(\frac{\beta(\nu+w)-\nu+m}{\nu}+1\right)^{\nu}} \tag{3.15}$$

and $I_{REG} = \int_0^{\infty}\left(\left(\frac{\beta(\nu+w)-\nu+m}{\nu}+1\right)g(x;\psi)[G(x;\psi)]^{\frac{\beta(\nu+w)-\nu+m}{\nu}}\right)^{\nu} dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{\beta(\nu+w)-\nu+m}{\nu}$. Furthermore, for $\delta > 0$

$$I_R(\nu) = (1-\nu)^{-1}\log\Bigg[\sum_{j=0}^{\infty} h_j \int_0^{\infty}\frac{\lambda^{\nu}\beta^{\nu}g^{\nu}(x;\psi)}{\overline{G}^{2\nu}(x;\psi)}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\nu(\beta-1)}$$

$$\times \quad \exp\left(-\lambda\nu\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j}dx\Bigg].$$

By considering the following expansions

$$\left[1-\exp\left(-\lambda\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)\right]^{j}=\sum_{p=0}^{\infty}(-1)^{p}\binom{j}{p}\exp\left(-\lambda p\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right),$$

$$\exp\left(-\lambda(\nu+p)\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta}\right)=\sum_{q=0}^{\infty}\frac{(-1)^{q}(-\lambda(\nu+p))^{q}}{q!}\left[\frac{G(x;\psi)}{\overline{G}(x;\psi)}\right]^{\beta q}$$

and

$$\overline{G}^{-(\beta(\nu+q)+\nu)}(x;\psi)=\sum_{m=o}^{\infty}\binom{-(\beta(\nu+q)+\nu)}{m}G(x;\psi),$$

we can write

$$\begin{aligned}
I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j,p,q,m=0}^{\infty}h_j\frac{\beta^{\nu}\lambda^{\nu}(-1)^{p+q+m}(-\lambda(\nu+p))^{q}}{q!}\\
&\times \binom{j}{p}\binom{-(\beta(\nu+q)+\nu)}{m}\frac{1}{\left(\frac{\beta(\nu+q)-\nu+m}{\nu}+1\right)^{\nu}}\\
&\times \int_0^{\infty}\left(\left(\frac{\beta(\nu+q)-\nu+m}{\nu}+1\right)g(x;\psi)[G(x;\psi)]^{\frac{\beta(\nu+q)-\nu+m}{\nu}}\right)^{\nu}dx\Bigg]\\
&= (1-\nu)^{-1}\log\Bigg[\sum_{j,p,q,m=0}^{\infty}h^{*}_{j,p,q,m}\exp(1-\nu)I_{REG}\Bigg], \qquad (3.16)
\end{aligned}$$

where

$$\begin{aligned}
h^{*}_{j,p,q,m} &= h_j\frac{\beta^{\nu}\lambda^{\nu}(-1)^{p+q}(-\lambda(\nu+p))^{q}}{q!}\\
&\times \binom{j}{p}\binom{-(\beta(\nu+q)+\nu)}{m}\frac{1}{\left(\frac{\beta(\nu+q)-\nu+m}{\nu}+1\right)^{\nu}} \qquad (3.17)
\end{aligned}$$

and $I_{REG}=\int_0^{\infty}\left(\left(\frac{\beta(\nu+q)-\nu+m}{\nu}+1\right)g(x;\psi)[G(x;\psi)]^{\frac{\beta(\nu+q)-\nu+m}{\nu}}\right)^{\nu}dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{\beta(\nu+q)-\nu+m}{\nu}$.

3.3. Moments and Generating Function

Let $X\sim MO-OW-G(\delta,\lambda,\beta,\psi)$, then the r^{th} moment can be obtained from equations (2.8) and (2.10). For $\delta\in(0,1)$,

$$E(X^{r})=\sum_{j,q,z,m=0}^{\infty}w^{*}_{j,q,z,m}E(W^{r}_{\beta(z+1)+m}),$$

where $w^*_{j,q,z,m}$ is as defined in equation (2.9) and $E(W^r_{\beta(z+1)+m})$ denotes the r^{th} moment of $W_{\beta(z+1)+m}$ which follows an Exp-G distribution with power parameter $(\beta(z+1)+m)$. For $\delta > 1$

$$E(X^r) = \sum_{j,p,w,b=0}^{\infty} v^*_{j,p,w,b} E(W^r_{\beta(w+1)+b}), \tag{3.18}$$

where $v^*_{j,p,w,b}$ is as defined in equation (2.11) and $E(W^r_{\beta(w+1)+b})$ denotes the r^{th} moment of $W_{\beta(w+1)+b}$ which follows an Exp-G distribution with power parameter $(\beta(w+1)+b)$. The incomplete moments can be obtained as follows:
For $\delta \in (0,1)$

$$I_X(t) = \int_0^t x^s f_{MO-OW-G}(x;\delta,\lambda,\beta,\psi)dx = \sum_{j,q,z,m=0}^{\infty} w^*_{j,q,z,m} I_{\beta(z+1)+m}(t),$$

where $I_{\beta(z+1)+m}(t) = \int_0^t x^r g_{\beta(z+1)+m}(x;\xi)dx$ and $w^*_{j,q,z,m}$ is as defined in equation (2.9). Also, For $\delta > 1$

$$I_X(t) = \int_0^t x^s f_{MO-OW-G}(x;\delta,\lambda,\beta,\psi)dx = \sum_{j,p,w,b=0}^{\infty} v^*_{j,p,w,b} I_{\beta(w+1)+b}(t),$$

where $I_{\beta(w+1)+b}(t) = \int_0^t x^r g_{\beta(w+1)+b}(x;\psi)dx$ and $v^*_{j,p,w,b}$ is as defined in equation (2.11). The moment generating function (mgf) of X is given by:
For $\delta \in (0,1)$

$$M_X(t) = \sum_{j,q,z,m=0}^{\infty} w^*_{j,q,z,m} E(e^{tW_{\beta(z+1)+m}}),$$

where $E(e^{tW_{\beta(z+1)+m}})$ is the mgf of the Exp-G distribution with power parameter $(\beta(z+1)+m)$ and $w^*_{j,q,z,m}$ is as defined in equation (2.9). For $\delta > 1$

$$M_X(t) = \sum_{j,p,w,b=0}^{\infty} v^*_{j,p,w,b} E(e^{tW_{\beta(w+1)+b}}),$$

where $E(e^{tW_{\beta(w+1)+b}})$ is the mgf of the Exp-G distribution with power parameter $(\beta(w+1)+b)$ and $v^*_{j,p,w,b}$ is as defined in equation (2.11). Furthermore, we can obtain the characteristic function given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$. For $\delta \in (0,1)$

$$\phi(t) = \sum_{j,q,z,m=0}^{\infty} w^*_{j,m,l} \phi_{\beta(z+1)+m}(t),$$

where $\phi_{\beta(z+1)+m}(t)$ is the characteristic function of Exp-G distribution with power parameter $(\beta(z+1)+m)$ and $w^*_{j,q,z,m}$ is as defined in equation (2.9). For $\delta > 1$

$$\phi(t) = \sum_{j,p,w,b=0}^{\infty} v^*_{j,p,w,b} \phi_{\beta(w+1)+b}(t),$$

where $\phi_{\beta(w+1)+b}(t)$ is the characteristic function of Exp-G distribution with power parameter $(\beta(w+1)+b)$ and $v^*_{j,p,w,b}$ is as defined in equation (2.11).

Table 2: Moments of the MO-OW-LLoG distribution for some parameter values

	(1.5,0.5,0.5,1)	(0.5,1.5,1,1.5)	(0.5,1.5,1,1.5)	(1,1.5,1,0.5)	(1.3,1.5,1.5,0.5)
E(X)	0.2889	0.2385	0.2385	0.1699	0.2228
E(X^2)	0.1876	0.1305	0.1305	0.0881	0.1262
E(X^3)	0.1371	0.0881	0.0881	0.0585	0.0868
E(X^4)	0.1074	0.0660	0.0660	0.0436	0.0659
E(X^5)	0.0880	0.0526	0.0526	0.0347	0.0530
SD	0.3227	0.2714	0.2714	0.2433	0.2766
CV	1.1170	1.1382	1.1382	1.4319	1.2412
CS	0.6762	1.0897	1.0897	1.6285	1.1637
CK	2.0293	3.0960	3.0960	4.7398	3.1908

The coefficients of variation (CV), skewness (CS) and kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu_r' - \sum_{s=1}^{r-1}\binom{r-1}{s-1}\mu_{r-s}'\kappa_s$, where $\mu_r' = E(X-\mu_1')^r$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$. A table of moments, SD, CV, CS, and CK for selected parameter values of the special case of the Marshall-Olkin Odd Weibull-log logistic (MO-OW-LLoG) distribution are given in Table 2.

4. Estimation

If $X_i \sim MO{-}OW{-}G(\beta,\lambda,\delta,\psi)$ with the parameter vector $\Delta = (\beta,\lambda,\delta,\psi)^T$. The log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell &= n\log\lambda + n\log\beta + n\log\delta + \sum_{i=1}^{n}\log[g(x_i;\psi)] + (\beta-1)\sum_{i=1}^{n}\log[G(x_i;\psi)] \\
&- \lambda\sum_{i=1}^{\infty}\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta} - (\beta+1)\sum_{i=1}^{n}\log\overline{G}(x_i;\psi) \\
&- 2\sum_{i=1}^{n}\log\left[1 - \overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right].
\end{aligned}$$

The score vector $U = \left(\frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\delta}\frac{\partial\ell}{\partial\psi_k}\right)$ has elements given by

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(G(x_i;\psi)) - \lambda\sum_{i=1}^{n}\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\log\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]$$

$$- \sum_{i=1}^{n} \log[\overline{G}(x_i;\psi)] - 2\lambda\delta \sum_{i=1}^{n} \left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta} \log\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]$$
$$\times \exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\left[1-\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right]^{-1},$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta} - 2\sum_{i=1}^{n}\left[\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right]$$
$$\times \left[1-\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right]^{-1},$$

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + 2\sum_{i=1}^{n}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\Bigg]\left[1-\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right]^{-1},$$

and

$$\frac{\partial \ell}{\partial \psi_k} = \sum_{i=1}^{n}\frac{1}{g(x_i;\psi)}\frac{\partial g(x_i;\psi)}{\partial \psi_k} + (\beta-1)\sum_{i=1}^{n}\frac{1}{G(x_i;\psi)}\frac{\partial G(x_i;\psi)}{\partial \psi_k} - \lambda\beta\sum_{i=1}^{n}\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta-1}$$
$$\times \frac{\partial}{\partial \psi_k}\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta} - (\beta+1)\sum_{i=1}^{n}\frac{1}{\overline{G}(x_i;\psi)}\frac{\partial \overline{G}(x_i;\psi)}{\partial \psi_k}$$
$$- 2\sum_{i=1}^{n}\left[1-\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right]^{-1}\frac{\partial}{\partial \psi_k}\left[1-\overline{\delta}\exp\left(-\lambda\left[\frac{G(x_i;\psi)}{\overline{G}(x_i;\psi)}\right]^{\beta}\right)\right],$$

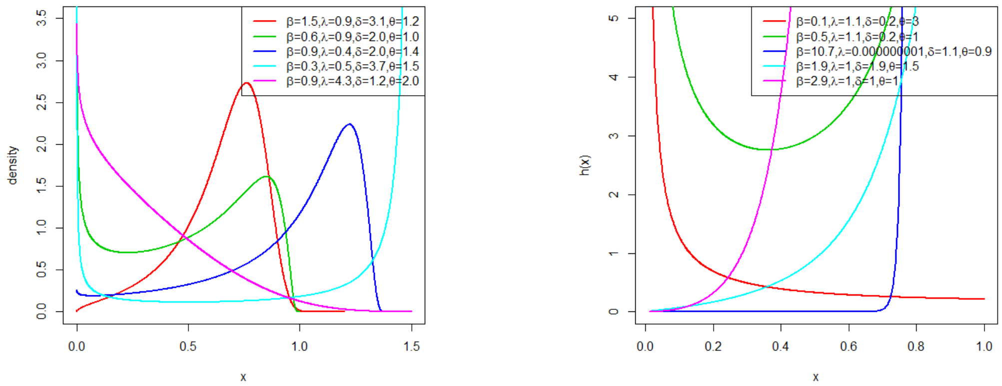
respectively. These partial derivatives are not in closed form and can be solved using R, MATLAB and SAS software by use of iterative methods. Furthermore, we use the Fisher information matrix to obtain estimates of the confidence intervals for the model parameters $\Delta = (\beta, \lambda, \delta, \psi)$. The Fisher information matrix is given by

$$J(\Delta) = \begin{pmatrix} J_{\beta\beta}(\Delta) & J_{\beta\lambda}(\Delta) & J_{\beta\delta}(\Delta) & J_{\beta\psi}(\Delta) \\ J_{\lambda\beta}(\Delta) & J_{\lambda\lambda}(\Delta) & J_{\lambda\delta}(\Delta) & J_{\lambda\psi}(\Delta) \\ J_{\delta\beta}(\Delta) & J_{\delta\lambda}(\Delta) & J_{\delta\delta}(\Delta) & J_{\delta\psi}(\Delta) \\ J_{\psi\beta}(\Delta) & J_{\psi\lambda}(\Delta) & J_{\psi\delta}(\Delta) & J_{\psi\psi}(\Delta) \end{pmatrix}, \tag{4.1}$$

where $J_{i,j} = \frac{-\partial^2 \ell(\Delta)}{\partial i \partial j}$, for $i, j = \beta, \lambda, \delta, \psi$, where ψ is a p component vector, $J_{\psi\psi}(\Delta)$ is a $p \times p$ matrix, $J_{\beta\psi}(\Delta)$, $J_{\lambda\psi}(\Delta)$ and $J_{\delta\psi}(\Delta)$ has $p \times 1$ components, respectively. Under the usual regularity conditions $\hat{\Delta}$ is asymptotically normal distributed, that is $\hat{\Delta} \sim N(\underline{0}, I^{-1}(\Delta))$ as $n \longrightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the information matrix evaluated at $\hat{\Delta}$.

5. Some Special Cases

In this section, we present some special cases of the MO-OW-G family of distributions. We restrict the baseline distribution to at most two parameter model so as to avoid the problem of over-parametrization.



(a) (b)

Figure 1: Plots of the pdf and hrf for the MO-OW-U distribution

5.1. Marshal-Olkin-Odd Weibull-Uniform (MO-OW-U) Distribution

Suppose the uniform distribution is the baseline distribution. The uniform distribution has pdf and cdf given by $g(x) = 1/\theta$ and $G(x, \theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the MO-OW-U distribution have the cdf and pdf given by

$$F_{MO-OW-U}(x;\beta,\delta,\lambda,\theta) = 1 - \left(\frac{\delta \exp\left[-\lambda \left(\frac{(x/\theta)}{1-(x/\theta)} \right)^{\beta} \right]}{1 - \left[(1-\delta) \exp\left[-\lambda \left(\frac{(x/\theta)}{1-(x/\theta)} \right)^{\beta} \right] \right]} \right)$$

and

$$f_{MO-OW-U}(x;\beta,\delta,\lambda,\theta) = \frac{\lambda\beta\delta(1/\theta)(x/\theta)^{\beta-1} \exp\left(-\lambda \left[\frac{(x/\theta)}{1-(x/\theta)} \right]^{\beta} \right)}{[1-(x/\theta)]^{\beta+1} \left(1 - \left[(1-\delta) \exp\left(-\lambda \left[\frac{(x/\theta)}{1-(x/\theta)} \right]^{\beta} \right) \right] \right)^2}$$

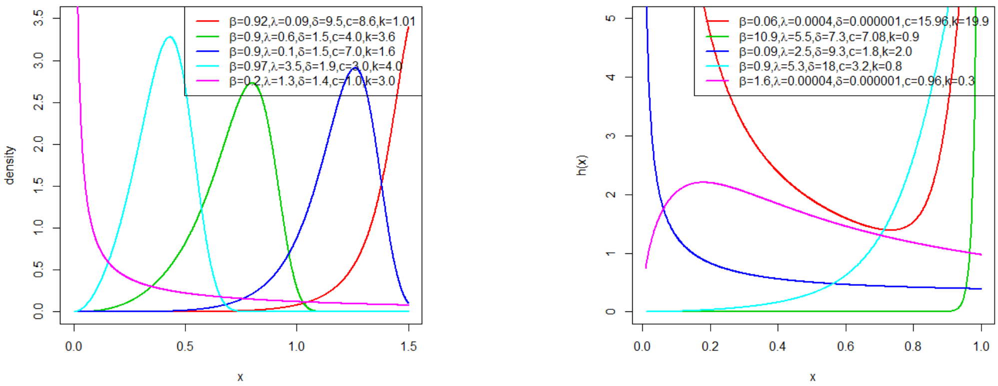
respectively, for $\beta, \lambda, \delta > 0$ and $0 < x < \theta$. The corresponding hazard rate function is given by

$$\begin{aligned} h_{MO-OW-U}(x;\beta,\delta,\lambda,\theta) &= \frac{f_{MO-OW-U}(x;\beta,\delta,\lambda,\theta)}{1 - F_{MO-OW-U}(x;\beta,\delta,\lambda,\theta)} \\ &= \frac{\lambda\beta(1/\theta)(x/\theta)^{\beta-1}}{[1-(x/\theta)]^{\beta+1} \left(1 - \left[(1-\delta) \exp\left(-\lambda \left[\frac{(x/\theta)}{1-(x/\theta)} \right]^{\beta} \right) \right] \right)} \end{aligned}$$

Figures 1 (a) and 1 (b) shows the plots of the pdfs and hrfs of MO-OW-U distribution for selected parameters values. The pdf can take various shapes including almost symmetric, right and left-skewed and U-shaped. Graphs of the hazard rate function exhibit increasing, bathtub, reverse J and J-shape.

5.2. Marshal-Olkin-Odd Weibull-Burr XII (MO-OW-BXII) Distribution

Consider the Burr XII distribution as the baseline distribution with pdf and cdf given by $g(x; c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}$ and $G(x; c, k) = 1 - (1 + x^c)^{-k}$, for $c, k > 0$, respectively. We can define the cdf and



(a) (b)

Figure 2: Plots of the pdf and hrf for the MO-OW-B distribution

pdf of the MO-OW-BXII distribution as

$$F_{MO-OW-BXII}(x;\beta,\delta,\lambda,c,k) = 1-\left(\frac{\delta\exp\left[-\lambda\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\beta}\right]}{1-\left[(1-\delta)\exp\left[-\lambda\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k})}\right)^{\beta}\right]\right]}\right)$$

and

$$\begin{aligned} f_{MO-OW-BXII}(x;\beta,\delta,\lambda,c,k) &= \frac{\lambda\beta\delta ckx^{c-1}(1+x^c)^{-(k+1)}(1-(1+x^c)^{-k})^{\beta-1}}{(1+x^c)^{-k(\beta+1)}} \\ &\times \frac{\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right]^{\beta}\right)}{\left(1-\left[(1-\delta)\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right]^{\beta}\right)\right]\right)^2} \end{aligned}$$

respectively, for $b,\alpha,\beta,c,k>0$. The MO-OW-Lomax and MO-OW-log logistic distributions are special cases of the MO-OW-BXII distribution when c=1 and k=1 respectively. The hazard rate function is given by

$$\begin{aligned} h_F(x;\beta,\delta,\lambda,c,k) &= \frac{\lambda\beta\delta ckx^{c-1}(1+x^c)^{-(k+1)}(1-(1+x^c)^{-k})^{\beta-1}}{(1+x^c)^{-k(\beta+1)}} \\ &\times \frac{1}{\left(1-\left[(1-\delta)\exp\left(-\lambda\left[\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right]^{\beta}\right)\right]\right)}. \end{aligned}$$

As shown in Figures 2 (a) and 2 (b) the MO-OW-BXII distribution has almost symmetric, right and left heavy tailed, J and reverse-J shaped pdfs as well as bathtub, increasing, upside bathtub, J and reversed-J shaped hrfs for different values of parameters. This implies that the MO-OW-BXII distribution can be very useful for fitting data sets with various shapes.

5.3. Marshal-Olkin-Odd Weibull-Kumaraswamy (MO-OW-K) Distribution

By taking the kumaraswamy distribution as the baseline distribution with pdf and cdf given by $g(x;a,b) = abx^{a-1}(1-x^a)^{\lambda-1}$ and $G(x;a,b) = 1-(1-x^a)^b$, for $x > 0$, a, and $b > 0$, respectively, we get the Marshal-Olkin-Odd Weibull-kumaraswamy (MO-OW-K) distribution with cdf and pdf given by

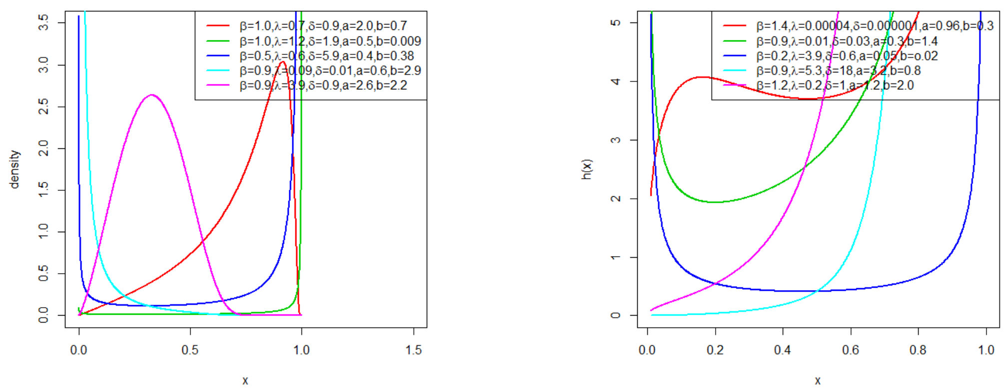
$$F_{MO-OW-K}(x;\beta,\delta,\lambda,a,b) = 1 - \left(\frac{\delta \exp\left[-\lambda \left(\frac{1-(1-x^a)^b}{(1-x^a)^b} \right)^{\beta} \right]}{1 - \left[(1-\delta) \exp\left[-\lambda \left(\frac{1-(1+x^a)^b}{(1-x^a)^b)} \right)^{\beta} \right] \right]} \right)$$

and

$$\begin{aligned} f_{MO-OW-K}(x;\beta,\delta,\lambda,a,b) &= \frac{\lambda\beta\delta abx^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\beta-1}}{(1-x^a)^{b(\beta+1)}} \\ &\times \frac{\exp\left(-\lambda \left[\frac{1-(1-x^a)^b}{(1-x^a)^b} \right]^{\beta} \right)}{\left(1 - \left[(1-\delta) \exp\left(-\lambda \left[\frac{1-(1-x^c)^b}{(1-x^a)^b} \right]^{\beta} \right) \right] \right)^2}, \end{aligned}$$

respectively, for $\beta, \lambda, \delta, a, b > 0$ and $x > 0$. The hazard rate function is given by

$$\begin{aligned} h_F(x;\beta,\delta,\lambda,a,b) &= \frac{\lambda\beta abx^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\beta-1}}{(1-x^a)^{b(\beta+1)}} \\ &\times \frac{1}{\left(1 - \left[(1-\delta) \exp\left(-\lambda \left[\frac{1-(1-x^c)^b}{(1-x^a)^b} \right]^{\beta} \right) \right] \right)}. \end{aligned}$$



(a) (b)

Figure 3: Plots of the pdf and hrf for the MO-OW-K distribution

Figures 3 (a) and 3 (b) shows the plots of the pdfs and hrfs of MO-OW-K distribution for selected parameters values. The pdf can take various shapes including U-shape, right skewed, left skewed and reverse-J. Graphs of the hazard rate function exhibit increasing, upside bathtub followed by bathtub, U-shape, and bathtub shape.

6. Simulation Study

In this subsection, we conduct Monte Carlo simulation study to assess the finite sample behaviour of the maximum likelihood estimates (MLEs) of β, λ, δ, and c. The exactness of the MLEs is examined by means of the Mean, RMSE and Average Bias. We generate N = 1000 samples of size n = 25, 50, 100, 200, 400, 800 and 1000 from simulations carried out using the R software. The exact parameter values used in the data generating process are I. $\beta = 1.5$, $\lambda = 0.1$, $\delta = 0.1$, $c = 0.5$, II. $\beta = 1.5$, $\lambda = 0.05$, $\delta = 0.1$, $c = 0.5$, III. $\beta = 1.5$, $\lambda = 0.05$, $\delta = 0.05$, $c = 0.5$ and IV $\beta = 1.5$, $\lambda = 1.5$, $\delta = 0.1$, $c = 0.5$. Simulation results are shown in Table 3. The results shows that as the sample size increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs and average bias decays toward zero for all the parameters. Note that the bias and RMSE for the estimated parameter, say, $\hat{\Delta}$, say, are given by:

$$Bias(\hat{\Delta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta) \quad \text{and} \quad RMSE(\hat{\Delta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta)^2}{N}},$$

respectively.

7. Applications

Three real data examples on the MO-OW-LLoG distribution are presented to assess the usefulness of the special case of the family of the distributions. We present the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC). Other goodness-of-fit statistics are also presented, namely, Cramer von Mises (W^*) and Andersen-Darling (A^*) (Chen and Balakrishnan [13]). These statistics are used to verify which model fits the given data well. The smaller the values of these statistics, the better the model. Kolmogorov-Smirnov (K-S) statistic, its p-value and sum of squares (SS) from probability plots were also used to assess the fit of the model. The model with the smaller KS value and the highest p-value for the K-S statistic, is regarded as the best fitting model.

The nlm function in R was used to estimate the model parameter. We present the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics. The results are shown in Tables 4, 5 and 6. Furthermore, we present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [12]). The plots are shown in Figures 4 (a), 4 (b), 5 (a), 5 (b) and 6 (a), 6 (b).

We compare the MO-OW-LLoG distribution to several competing non-nested distributions with equal number of parameters. The non-nested models considered are the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [25], Topp-Leone-Weibull-Lomax (TL-WLx) by Jamal et al. [24], Beta-Weibull (BW) by Cordeiro et al. [16], Kumaraswamy-Weibull (KwW) by Cordeiro et al. [17], odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. [1], beta odd Lindley-Uniform (BOL-U) and beta odd Lindley-Exponential (BOL-E) by Chipepa et al. [14]. The pdfs of the non-nested models are as follows:

$$\begin{aligned} f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda) &= \frac{\alpha\theta^2\gamma\lambda^\gamma x^{\gamma-1}e^{-(\lambda x)^\gamma}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^\gamma})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma})^{\theta-1}} \\ &\times \left(1-\beta\log\left[\frac{e^{-(\lambda x)^\gamma}}{(1-e^{-(\lambda x)^\gamma})^\alpha + e^{-\alpha(\lambda x)^\gamma}}\right]\right), \end{aligned}$$

for $\alpha, \beta, \gamma, \theta, \lambda > 0$,

$$\begin{aligned} f_{TL-WLx}(x;a,b,\alpha,\theta) &= 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1} \\ &\times \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right) \end{aligned}$$

Table 3: Monte Carlo Simulation Results for MO-OW-LLoG Distribution: Mean, RMSE and Average Bias

	I	$\beta = 1.5,\ \lambda = 0.1,\ \delta = 0.1,\ c = 0.5$			II $\beta = 1.5,\ \lambda = 0.05,\ \delta = 0.1,\ c = 0.5$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
β	25	1.455484	0.154459	-0.044516	1.460361	0.130660	-0.039639
	50	1.478000	0.112891	-0.022000	1.478770	0.092015	-0.021230
	100	1.491259	0.046942	-0.008741	1.490011	0.061837	-0.009989
	200	1.494417	0.016532	-0.005583	1.494440	0.015894	-0.005560
	400	1.497048	0.011167	-0.002953	1.496843	0.010890	-0.003157
	800	1.497630	0.008756	-0.002370	1.497507	0.008835	-0.002493
	1000	1.498136	0.007220	-0.001864	1.497752	0.007291	-0.002248
λ	25	0.390615	0.677934	0.290615	0.223070	0.416473	0.173070
	50	0.231149	0.420525	0.131149	0.135306	0.301292	0.085306
	100	0.159564	0.178416	0.059564	0.091579	0.178652	0.041579
	200	0.132074	0.084368	0.032074	0.070444	0.049643	0.020444
	400	0.111342	0.045592	0.011342	0.058376	0.026675	0.008376
	800	0.108895	0.033758	0.008895	0.055842	0.019325	0.005842
	1000	0.105198	0.027484	0.005198	0.054046	0.015749	0.004046
δ	25	1.742389	9.936280	1.642389	1.071353	5.616407	0.971353
	50	0.849665	7.480301	0.749665	0.766570	7.187854	0.666570
	100	0.215731	1.098310	0.115731	0.340447	3.489555	0.240447
	200	0.142652	0.107351	0.042652	0.145693	0.110129	0.045693
	400	0.115353	0.053938	0.015353	0.118042	0.056374	0.018042
	800	0.110857	0.038790	0.010857	0.111945	0.040363	0.011945
	1000	0.106471	0.030934	0.006471	0.108126	0.032208	0.008126
c	25	0.479960	0.093164	-0.020040	0.473725	0.082536	-0.026275
	50	0.489383	0.074201	-0.010617	0.480913	0.062193	-0.019087
	100	0.488970	0.046781	-0.011030	0.486775	0.047394	-0.013225
	200	0.489522	0.034612	-0.010478	0.487764	0.035519	-0.012236
	400	0.494346	0.023590	-0.005654	0.492743	0.024093	-0.007257
	800	0.495253	0.017729	-0.004748	0.494355	0.017876	-0.005645
	1000	0.496287	0.014867	-0.003713	0.494890	0.015705	-0.005110
	III	$\beta = 1.5,\ \lambda = 0.05,\ \delta = 0.05,\ c = 0.5$			IV $\beta = 1.5,\ \lambda = 1.5,\ \delta = 0.1,\ c = 0.5$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
β	25	1.377781	0.376780	-0.122219	1.456154	0.203980	-0.043846
	50	1.432019	0.197844	-0.067981	1.465122	0.170934	-0.034878
	100	1.465167	0.122942	-0.034833	1.467777	0.155349	-0.032223
	200	1.478920	0.083648	-0.021080	1.471769	0.119541	-0.028231
	400	1.493400	0.048118	-0.006600	1.490315	0.052075	-0.009685
	800	1.494981	0.032451	-0.005019	1.492888	0.025617	-0.007112
	1000	1.496166	0.028227	-0.003835	1.493005	0.030360	-0.006995
λ	25	3.114495	2.759841	1.614494	2.786741	4.188283	1.286741
	50	2.353414	1.737693	0.853414	2.242029	2.925118	0.742029
	100	1.969639	1.101005	0.469639	2.052543	2.244073	0.552543
	200	1.792233	0.788281	0.292233	1.852549	1.343353	0.352549
	400	1.622939	0.488766	0.122939	1.644840	0.654835	0.144840
	800	1.575587	0.348244	0.075587	1.617129	0.471413	0.117129
	1000	1.549499	0.309251	0.049499	1.589252	0.365110	0.089252
δ	25	1.872018	12.738300	1.772018	0.104715	1.374978	0.094715
	50	1.169872	9.785217	1.069872	0.035133	0.143737	0.025133
	100	0.307460	3.174167	0.207460	0.017603	0.035817	0.007603
	200	0.142418	0.109792	0.042418	0.014156	0.015845	0.004156
	400	0.117115	0.057684	0.017115	0.011659	0.005796	0.001659
	800	0.109607	0.038688	0.009607	0.011199	0.003938	0.001199
	1000	0.106402	0.032380	0.006402	0.010984	0.003321	0.000984
c	25	0.930601	0.203814	0.030601	0.946265	0.360209	0.046265
	50	0.915867	0.114666	0.015867	0.930862	0.251683	0.030862
	100	0.911880	0.073479	0.011880	0.936683	0.212918	0.036683
	200	0.904338	0.051354	0.004338	0.920304	0.132201	0.020304
	400	0.901570	0.036275	0.001570	0.902860	0.052817	0.002860
	800	0.900504	0.026092	0.000504	0.900113	0.020590	0.000113
	1000	0.901268	0.023539	0.001268	0.901169	0.021173	0.001169

$$\times \quad \left[1-\exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)\right]^{\theta-1},$$

for $a, b, \alpha, \theta > 0$,

$$f_{BW}(x; a, b, \alpha, \beta) = \frac{\beta\alpha^{\beta}}{B(a,b)} x^{\beta-1} e^{-b(\alpha x)^{\beta}} (1 - e^{-(\alpha x)^{\beta}})^{a-1},$$

for $a, b, \alpha, \beta > 0$,

$$f_{KwW}(x; a, b, \alpha, \beta) = ab\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}} (1 - e^{-(\alpha x)^{\beta}})^{a-1} (1 - (1 - e^{-(\alpha x)^{\beta}})^{a})^{b-1},$$

for $a, b, \alpha, \beta > 0$,

$$\begin{aligned} f_{BOL-U}(x; a, b, \lambda, \theta) &= \frac{1}{B(a,b)} \left[1 - \frac{\lambda + (1 - x/\theta)}{(1+\lambda)(1 - x/\theta)} \exp\left\{-\lambda \frac{x}{(\theta - x)}\right\}\right]^{a-1} \\ &\times \left[\frac{\lambda + (1 - x/\theta)}{(1+\lambda)(1 - x/\theta)} \exp\left\{-\lambda \frac{x}{(\theta - x)}\right\}\right]^{b-1} \\ &\times \frac{\lambda^2}{(1+\lambda)} \frac{\theta^2}{(\theta - x)^3} \exp\left\{-\lambda \frac{x}{(\theta - x)}\right\}, \end{aligned}$$

for $a, b, \lambda > 0$ and $0 < x < \frac{1}{\theta}$,

$$\begin{aligned} f_{BOL-E}(x; a, b, \lambda, \theta) &= \frac{1}{B(a,b)} \left[1 - \frac{\lambda + e^{-\theta x}}{(1+\lambda)e^{-\theta x}} \exp\left\{-\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{a-1} \\ &\times \left[\frac{\lambda + e^{-\theta x}}{(1+\lambda)e^{-\theta x}} \exp\left\{-\lambda \frac{(1 - e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{b-1} \\ &\times \frac{\lambda^2}{(1+\lambda)} \frac{(\theta e^{-\theta x})}{e^{-3\theta x}} \exp\left\{-\lambda \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}, \end{aligned}$$

for $a, b, \lambda, \theta > 0$ and

$$f_{OLLEW}(x; \alpha, \beta, \gamma, \theta) = \frac{\theta\beta\gamma x^{\beta-1} e^{-(x/\alpha)^{\beta}} [1 - e^{-(x/\alpha)^{\beta}}]^{\gamma\theta-1} (1 - [1 - e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta-1}}{\alpha\beta([1 - e^{-(x/\alpha)^{\beta}}]^{\theta\gamma} + (1 - [1 - e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta})^2},$$

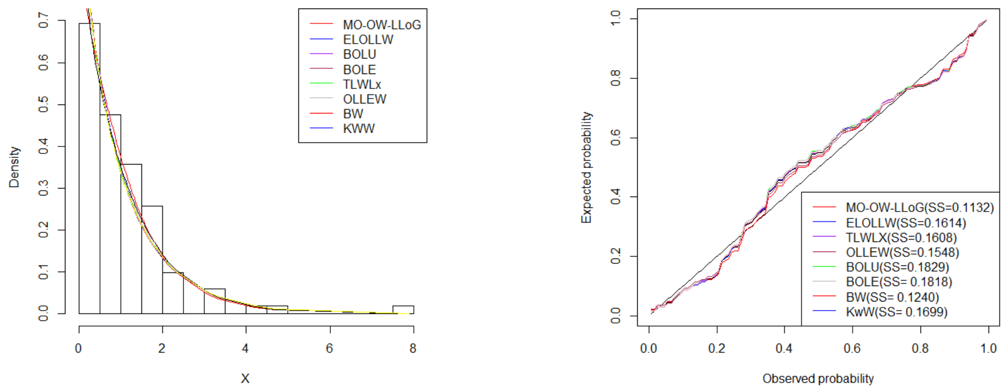
for $a, b, \lambda, \gamma, \theta > 0$ and the ELOLLW distribution we considered a case when $\alpha = 1$.

7.1. Kevlar 49/Epoxy Strands Failure at 90% Data

We fit the MO-OW-LLoG distribution to the data set reported by Andrews and Herzberg [7] and also by Barlow, Toland and Freeman [9]. The data represents failure times (in hours) of kevlar 49/epoxy strands subjected to constant sustained pressure at the 90% stress level. The observations are as follows: 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

Table 4: Parameter estimates and goodness of fit statistics for various models fitted for Kevlar data set

	Estimates				Statistics							
Model	β	λ	δ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-OW-LLoG	0.8068	1.9955	3.9163	0.8597	203.7	211.7	212.1	222.1	0.1190	0.7369	0.0704	0.6985
	(0.1052)	(0.8384)	(3.8563)	(0.0984)								
	β	λ	θ	γ								
ELOLLW	7.06921	0.15560	7.39364	0.83740	205.2	213.2	213.6	223.7	0.1666	0.9601	0.0816	0.5123
	(4.1287)	(0.1115)	(3.9446)	(0.1094)								
	a	b	α	θ								
TLWLX	0.6616	0.7096	1.3632	0.6079	205.4	213.4	213.8	223.9	0.1625	0.9447	0.0835	0.4824
	(0.6596)	(0.9640)	(0.9470)	(0.4456)								
	α	β	γ	θ								
OLLEW	1.6251	1.1121	0.6081	1.1687	205.5	213.5	214.0	224.0	0.1616	0.9432	0.0798	0.5408
	(2.6163)	(0.4424)	(0.9012)	(0.9463)								
	a	b	λ	θ								
BOL-U	8.7181×10^{1}	1.0323×10^{-1}	2.3456×10^{4}	2.8294×10^{5}	205.7	213.7	214.1	224.1	0.1820	1.0333	0.0893	0.394
	(1.0675×10^{-1})	(1.6271)	(7.1192×10^{-4})	(5.9016×10^{-5})								
	a	b	λ	γ								
BOL-E	8.7078×10^{-1}	3.0492×10^{1}	2.0405×10^{1}	1.4327×10^{-3}	205.7	213.7	214.1	224.1	0.1814	1.0308	0.089	0.3973
	(1.0643×10^{-1})	(4.9112×10^{-6})	(7.8702×10^{-6})	(2.3001×10^{-5})								
	a	b	λ	k								
BW	0.7077	0.1531	5.2437	1.0548	204.8	212.8	213.2	223.3	0.4673	3.3331	0.0723	0.6662
	(0.2179)	(0.1538)	(5.7751)	(0.1488)								
	a	b	α	β								
KwW	0.7790	3.8943	0.1957	1.1294	205.8	213.8	214.2	224.2	0.1791	1.0218	0.0847	0.4627
	(0.5151)	(3.1504)	(0.1636)	(0.6097)								



(a) (b)

Figure 4: Fitted densities and probability plots for kevlar data

The estimated variance-covariance matrix for the MO-OW-LLoG distribution is given by

$$\begin{bmatrix} 0.01106 & -0.08325 & -0.37946 & 0.01039 \\ -0.08325 & 0.70288 & 3.18199 & -0.07817 \\ -0.37946 & 3.18199 & 14.87083 & -0.35634 \\ 0.01039 & -0.07817 & -0.35634 & 0.00976 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\beta \in [0.8068 \pm 0.2062]$, $\lambda \in [1.9955 \pm 1.6432]$, $\delta \in [3.9163 \pm 7.5583]$ and $c \in [0.8592 \pm 0.1936]$.

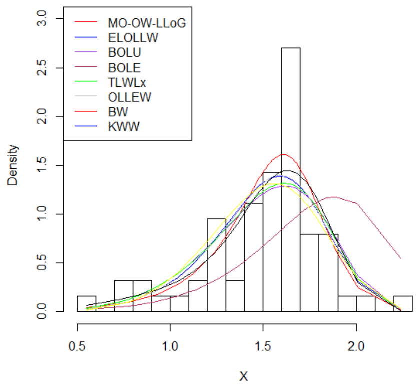
From the results presented in Table 4, we observe that the MO-OW-LLoG model has the lowest values for the goodness-of-fit statistics and the highest P-value for the K-S statistic compared to the non-nested models considered in this paper. Therefore, we conclude that the MO-OW-LLoG model fit the Kevlar data set better than the non-nested models ELLOW, OLLEW, BOL-E, BOL-U, TLWLX, KwW and BW distributions. Also, Figures 4 (a) and 4 (b) show that our proposed model performs better than the competing non-nested models on kevlar data.

7.2. Strengths of 1.5 cm Glass Fibres Data

The second data set is on strengths of 1.5 cm glass fibres. The data set was also analyzed by Bourguignon et al. [11]. The data are 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 5: Parameter estimates and goodness of fit statistics for various models fitted for glass fibres data set

	Estimates				Statistics							
Model	β	λ	δ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-OW-LLoG	2.2166	0.6948	16.6435	1.4445	24.1	32.1	32.8	40.6	0.1058	0.5912	0.1000	0.5548
	(0.1957)	(0.5651)	(20.7770)	(0.3001)								
	β	λ	θ	γ								
ELOLLW	0.8916	0.7976	0.4768	4.9441	28.4	36.4	37.1	45.0	0.1953	1.0753	0.1367	0.1896
	(0.2015)	(0.1531)	(0.3260)	(0.6592)								
	a	b	α	θ								
TLWLx	3.2597	0.1265	5.5684	0.7639	29.1	37.1	37.8	45.7	0.1934	1.0772	0.1433	0.1504
	(3.0637)	(0.1314)	(1.5082)	(0.2996)								
	α	β	γ	θ								
OLLEW	1.9920	8.7485	0.3021	1.6872	28.0	36.0	36.7	44.6	0.1864	1.0314	0.132	0.2223
	(0.2975)	(3.9396)	(0.2668)	(0.7436)								
	a	b	λ	θ								
BOLU	3.7723	9.3006×10^{1}	1.7071×10^{-1}	3.0137	29.9	37.9	38.6	46.5	0.2022	1.1294	0.1425	0.155
	(1.1920)	(9.1329×10^{-4})	(5.5793×10^{-2})	(2.9634×10^{-1})								
	a	b	λ	γ								
BOL-E	1.2000	7.0576	0.0277	1.9749	28.1	36.1	36.8	44.7	0.1550	0.8753	0.1301	0.2367
	(0.6328)	(0.0311)	(0.0355)	(0.5750)								
	a	b	λ	k								
BW	0.4493	0.0496	0.9199	7.0127	26.1	34.1	34.7	46.7	2.0396	11.1272	0.1133	0.3942
	(0.1819)	(0.0470)	(0.1499)	(0.8894)								
	a	b	α	β								
KwW	7.3919	4.7793×10^{4}	1.3586×10^{-1}	8.7760×10^{-1}	31.2	39.2	39.9	47.7	0.2563	1.4056	0.16346	0.0693
	(2.6561)	(3.3847×10^{-5})	(4.4551×10^{-2})	(2.4139×10^{-1})								



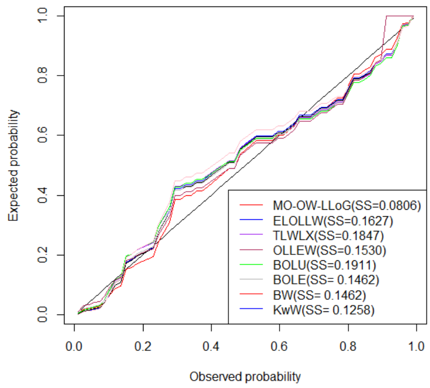


Figure 5: Fitted densities and probability plots for glass fibre data

The estimated variance-covariance matrix for the MO-OW-LLoG distribution is given by

$$\begin{bmatrix} 0.03830 & -0.10901 & -3.74280 & 0.05872 \\ -0.10901 & 0.31934 & 11.34569 & -0.16714 \\ -3.74280 & 11.34569 & 431.68213 & -5.73798 \\ 0.05872 & -0.16714 & -5.73798 & 0.09004 \end{bmatrix}$$

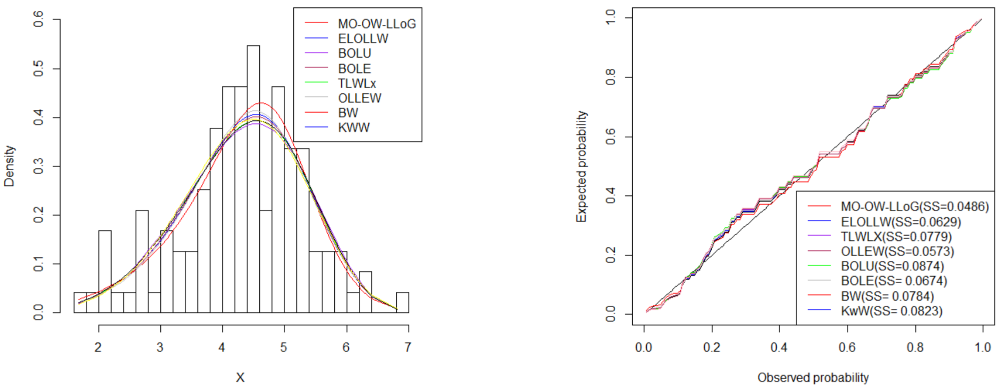
and the 95% confidence intervals for the model parameters are given by $\beta \in [2.2166 \pm 0.3836]$, $\lambda \in [0.69483 \pm 1.1076]$, $\delta \in [16.643481 \pm 40.7228]$ and $c \in [1.44446 \pm 0.5881]$. From the results presented in Table 5, we conclude that the MO-OW-LLoG model fit the glass fibres data better than the non-nested ELLOW, OLLEW, BOL-E, BOL-U, TLWLX, KwW and BW distributions. Also, Figures 5 (a) and 5 (b) show that our proposed model performs better than the competing models on strength of glass fibres data

7.3. Silicon Nitride Data

The third data set represent fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data set was analyzed by Nadarajah and Kotz [28] and also by Chipepa et al. [15]. The data are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

Table 6: Parameter estimates and goodness of fit statistics for various models fitted for Growth hormone data set

	Estimates				Statistics							
Model	β	λ	δ	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-OW-LLoG	2.7113	0.0134	5.6175	1.2265	335.6	343.6	344.0	354.7	0.0506	0.3047	0.0523	0.9009
	(0.0365)	(0.0058)	(0.0044)	(0.0806)								
	β	λ	θ	γ								
ELOLLW	1.0558	0.2683	0.6356	4.2260	336.4	344.4	344.8	355.6	0.0698	0.4309	0.0628	0.7363
	(0.3108)	(0.0703)	(0.5092)	(0.5822)								
	a	b	α	θ								
TLWLx	1.4695	0.1100	4.8479	0.8960	337.1	345.1	345.5	356.3	0.0828	0.5025	0.0692	0.6182
	(1.8065)	(0.1677)	(1.9954)	(0.3688)								
	α	β	γ	θ								
OLLEW	5.9243	6.8953	0.3277	1.7008	336.3	344.3	344.6	355.4	0.0646	0.4062	0.0599	0.7862
	(0.2975)	(3.9396)	(0.2668)	(0.7436)								
	a	b	λ	θ								
BOLU	4.3428	99.1503	0.2275	10.3191	337.7	345.7	346.1	356.9	0.0929	0.5615	0.0721	0.5651
	(1.1991)	(0.0017)	(0.0714)	(1.2905)								
	a	b	λ	γ								
BOL-E	2.3253	16.0976	0.1096	0.3817	337.1	345.1	345.5	356.2	0.0712	0.4406	0.0648	0.699
	(1.1065)	(0.0085)	(0.0887)	(0.1081)								
	a	b	λ	k								
BW	0.8013	12.3330	0.1299	5.6883	337.1	345.1	345.5	356.2	0.0830	0.5023	0.0695	0.6141
	(0.3130)	(5.2166×10^{-4})	(5.6038×10^{-3})	(1.4291)								
	a	b	α	β								
KwW	0.9574	447.3400	0.0062	5.1840	337.4	345.4	345.8	356.5	0.0919	0.5636	0.0722	0.5647
	(1.0265)	(5.5241×10^{-4})	(5.2846×10^{-3})	(5.5238)								



(a) (b)

Figure 6: Fitted densities and probability plots for silicon nitrate data

The estimated variance-covariance matrix for MO-OW-LLoG model on silicon nitrate data set is given by

$$\begin{bmatrix} 0.00133 & -0.00021 & 0.00016 & 0.00294 \\ -0.00021 & 0.00003 & -0.00003 & -0.00047 \\ 0.00016 & -0.00003 & 0.00002 & 0.00035 \\ 0.00294 & -0.00047 & 0.00035 & 0.00650 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\beta \in [2.7113 \pm 0.0715]$, $\lambda \in [0.0134 \pm 0.0114]$, $\delta \in [5.6175 \pm 0.0086]$ and $c \in [1.2265 \pm 0.1580]$.

Based on results shown in Table 6, we observe that the MO-OW-LLoG has the lowest values for the goodness-of-fit statistics and the highest p-value for the K-S statistic compared to the non-nested models. Thus, we conclude that the MO-OW-LLoG model fits the silicon nitrate data better than the non-nested models ELLOW, OLLEW, BOL-E, BOL-U, TLWLX, KwW and BW distributions. Also, Figures 6 (a) and 6 (b) show that our proposed model performs better than the competing non-nested models on silicon nitrate.

8. Concluding Remarks

We presented a new family of distributions, referred to as the MO-OW-G family of distributions. The new family of distributions can handle heavy tailed data and also accomodates monotonic and non-monotonic hazard rate shapes. The new family of distributions has a desirable tractability property and can be expressed as a linear combination of the Exp-G distribution. The applications provided showed that Marshal-Olkin Odd Weibull - LLoG distribution performs better than several non-nested models considered in this paper as shown in Tables 4, 5 and 6.

References

[1] Afify, A. Z., Alizadeh, M., Zayed, M., Ramires, T. G. and Louzada, F. (2018). The odd log-logistic exponentiated Weibull distribution: regression modelling, properties, and applications. Iranian Journal of Science and Technology, 42(4): 2273-2288.

[2] Alizadeh, A, Cordeiro, G. M., Brito, E., Demétrio, C. G (2015). The beta Marshall-Olkin family of distributions. Journal of Statistical Distributions and Applications, 4(2): 1–18.

[3] Alizadeh, M., Bagheri, S., Bahrami S. E., Ghobadi, S., and Nadarajah, S., (2017). Exponentiated power Lindley power series class of distribution: theory and applications. Communications in Statistics-Simulation and Computation, 47(9): 2499-2531.

[4] Alizadeh, A., Emadi, M., Doostparast, M., Cordeiro, G. M., Ortega, E., Pescim, R. (2015). A new family of distributions: The Kumaraswamy odd log-logistic, properties and applications. Hacettepe Journal of Mathematics and Statistics, 44: 1491-1512.

[5] Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M.Q. (2016). Topp-Leone family of distributions: some properties and application. Pakistan Journal of Statistics and Operation Research, 12(3): 443-451.

[6] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71(1): 63-79.

[7] Andrews, D. F. and Herzberg, A. M. (2012). Data: A collection of problems from many fields for the student and research worker. Springer Science & Business Media.

[8] Barreto-Souza, W., Lemonte, A.J. and Cordeiro, G.M. (2013). General results for the Marshall and Olkin's family of distributions. Annals of the Brazilian Academy of Sciences, 85(1): 3-21.

[9] Barlow, R. E., Toland, R. H., and Freeman, T. (1984). A Bayesian analysis of stress-rupture life of Kevlar/Epoxy spherical pressure vessels. Proceedings of the Canadian Conference in Applied Statistics, Edited by: Dwivedi, T. D. New York: Marcel Dekker.

[10] Bekker, R., and Mostert, A., (1988). Heterogeneity in survival analysis. Statistics in Medicine, 7: 1121-1137.

[11] Bourguignon, M., Silva R. B. and Cordeiro G. M. (2014).The Weibull-G family of probability distributions. Journal of Data Science, 12: 53-68.

[12] Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). Graphical methods of data analysis, Chapman and Hall.

[13] Chen, G. and Balakrishnan, N. (1985). A general purpose approximate goodness-of-fit test. Journal of Quality Technology, 27(2): 154-161.

[14] Chipepa, F., Oluyede, B. and Makubate, B. and Fagbamigbe A. F. (2019). The beta odd Lindley-G family of distributions with applications. Journal of Probability and Statistical Science, 17(1): 51-83.

[15] Chipepa, F., Oluyede, B. and Makubate, B. (2020). The Topp-Leone-Marshall-Olkin-G family of distributions with applications. International Journal of Statistics and Probability, 9(4): 15-32. doi:10.5539/ijsp.v9n4p15.

[16] Cordeiro, G. M., Gomes, A., da Silva, C. and Ortega, E. M. M. (2013). The beta exponentiated Weibull distribution. Journal of Statistical Computation and Simulations, 38(1), 114-138.

[17] Cordeiro, G. M. Ortega, E. M. M. and Nadarajaah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute, 347: 1399 -1429.

[18] Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81(7): 883-898.

[19] Cordeiro, G. M., Ortega, E. M., and da Cunha, D. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11: 1–27.

[20] Cordeiro, G. M., Alizadeh, M., Tahir, M. H., Mansoor, M., Bourguignon, M. and Hamedani, G. G. (2015). The beta odd log-logistic family of distributions. Hacettepe Journal of Mathematics and Statistics, HJMS.20157311545.

[21] Cordeiro, G.M., Alizadeh, M., Ozel, G., Hosseini, B., Ortega, E.M.M. and Altun, E. (2017). The generalized odd log-logistic family of distributions: properties, regression models and applications, Journal of Statistical Computation and Simulation, 87: 908-932.

[22] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and Methods, 31: 497-512.

[23] Gleaton, J. U. and Lynch, J. D. (2004). On the distribution of the breaking strain of a bundle of brittle elastic fibers. Advances in Applied Probability, 36: 98–115.

[24] Jamal, F., Reyad, H. M., Nasir, M. A., Chesneau, C., Shah, M. A. A. and Ahmed S. O. (2019). Topp-Leone Weibull-Lomax distribution: properties, regression model and applications. Hal-02270561.

[25] Korkmaz, M. C., Yousof, H. M. and Hamedani, G. G. (2018). The exponential Lindley odd log-logistic-G family: properties, characterizations and applications. Journal of Statistical Theory and Applications, 17(3): 554-571.

[26] Lepetu, L., Oluyede, B. O, Makubate, B., Foya, S., Mdlongwa, P (2017). Marshall-Olkin Log-logistic Extended Weibull: Theory, Properties and Applications. Journal of Data Science, 15: 691-722.

[27] Marshall A.N. and Olkin I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika, 84: 641-652.

[28] Nadarajah, S. and Kotz, S. (2007). On the alternative to Weibull function, Engineering Fracture Mechanics, 74(3): 451-456.

[29] Nofal, Z. M., A fy, A. Z., Yousof, H. M. and Cordeiro, G. M. (2017). The generalized transmuted-G family of distributions. Communications in Statistics - Theory and Methods, 46: 4119–4136.

[30] Oluyede, B. O., Huang, S. and Yang, T. (2015) A new class of generalized modified Weibull distribution with applications, Austrian Journal of Statistics 44: 45-68

[31] Pham, H., and Lai, C. D. (2007). On recent generalizations of the Weibull distribution. IEEE, Transaction on Reliability, 56: 454-458.

[32] Rényi, A. (1961). On measures of entropy and information. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. The Regents of the University of California, 1: 547-561.

[33] Shannon C. E. (1951). Prediction and entropy of printed english . The Bell System Technical Journal, 30(1): 50-64.

[34] Shaw, W. T. and Buckley, I. R. C. (2009). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv: 0901.0434.

[35] Torabi, H., Montazeri, N. H. (2014). The logistic-uniform distribution and its application. Communications in Statistics - Simulation and Computation, 43: 2551-2569.

[36] Zografos, K. and Balakrishnan, N. (2009). On families of generalized gamma-generated distributions and associated inference. Statistical Methodology, 6: 344-362.