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The Half Logistic Log-logistic Weibull Distribution: Model, Properties and Applications

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Abstract. A new three parameter distribution named Half Logistic Log-Logistic Weibull (HLLLW) distribution is developed. This model includes Half Logistic Log-Logistic (HLLL) distribution, Half Logistic Log-Logistic Exponential (HLLLE) distribution and Half Logistic Log-Logistic Rayleigh (HLLLR) distribution as sub-models. Structural properties including shapes, hazard function, reverse hazard function, quantile function, moments, conditional moments, mean deviations, Bonferroni and Lorenz Curves, Rényi entropy and distribution of order statistics are presented. We adopt the maximum likelihood method to estimate model parameters. To check the accuracy of the maximum likelihood estimates, various simulations were performed for different parameter settings and sample sizes. Finally, numerical examples are provided to test the goodness of fit of the proposed model compared to other models.
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1. Introduction

The log-logistic and Weibull distributions are widely used in several areas including reliability, economic, finance as well as actuarial sciences. However, these models have a limited range of behaviour and do not provide adequate fit to all the practical situations. The log-logistic distribution (known as the Fisk distribution) plays an important role in income. It is the probability distribution of a random variable whose logarithm has logistic distribution. Some generalization of the log-logistic model have appeared in the literature. For example, beta log-logistic distribution presented by Lemonte [20] and the log-logistic Weibull distribution by Oluyede et al. [26].

There are several extensions of the Weibull model which are able to depict more complex hazard rates like upside-down bathtub-shaped or unimodal shapes. Generalizations of the Weibull distribution in the literature include work by Xie et al. [30], Bebbington et al. [8], Cordeiro et al. [13] and Silva et al. [29]. Lai et al. [19] presented modification of Weibull distribution. Cordeiro et al. [11] tackled Lindley-Weibull and its applications. Lai et al. [18] studied modified Weibull distribution. Famoye et al. [16] developed the beta Weibull distribution. Pal et al. [27] introduced the exponentiated Weibull distribution. Mudholkar and Srivastata [21] developed the generalized Weibull distribution, Ibrahim and Yousof [17] presented the transmuted Topp-Leone Weibull

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lifetime distribution, Afify et al. [1] introduced Marshall–Olkin power generalized Weibull distribution, Okasha et al. [25] developed Marshall–Olkin extended inverse Weibull distribution, Ahmed et al. [2] developed the Topp-Leone Marshall Olkin-Weibull distribution, Aryal et al. [5] presented the Topp-Leone generated Weibull distribution, Nofal et al. [24] developed the Kumaraswamy transmuted exponentiated additive Weibull distribution.

In this paper, we study a new three parameter model by combining the half logistic and the log-logistic Weibull distributions called Half-Logistic Log-Logistic Weibull (HLLLW) distribution. Half-Logistic transformation was applied to several well known distributions. Anwar and Zahoor [4] introduced the half-logistic Lomax distribution for lifetime modeling, Anwar and Bibi [3] developed the half-logistic generalized Weibull distribution. Muhammad and Yahaya [22] established the half logistic-Poisson distribution.

Cordeiro et al. [12] define the cumulative distribution function (cdf) of the new type I half-logistic-G(TIHL-G) family of distributions by

$$\begin{aligned} F(x;\lambda,\underline{\psi}) &= \int_0^{-ln\left(1-G\left(x;\underline{\psi}\right)\right)} \frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2}dt \\ &= \frac{1-\left[1-G(x;\underline{\psi})\right]^{\lambda}}{1+\left[1-G(x;\underline{\psi})\right]^{\lambda}}, \end{aligned} \tag{1.1}$$

where $G(x;\underline{\psi})$ is the baseline cdf depending on a parameter vector $\underline{\psi}$ and $\lambda > 0$ is an additional shape parameter. If we take $\lambda = 1$, then the TIHL-G reduces to the half logistic-G (HL-G) distribution with cdf

$$F(x;\underline{\psi}) = \frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}, \tag{1.2}$$

where $\overline{G}(x;\underline{\psi}) = 1 - G(x;\underline{\psi})$.

The corresponding probability density function (pdf) to (1.2) is given by

$$f(x;\underline{\psi}) = \frac{2g(x;\underline{\psi})}{\left[1+\overline{G}(x;\underline{\psi})\right]^2}, \tag{1.3}$$

where $g(x;\underline{\psi}) = \frac{dG(x;\underline{\psi})}{dx}$ is the baseline pdf.

The three parameter cdf of the log-logistic Weibull (LLW) distribution (Oluyede et al. [26]) is given by

$$G_(x;\alpha,\beta,c) = 1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}, \quad \text{for } x>0, \text{ and } \alpha,\ \beta\ , c>0. \tag{1.4}$$

The corresponding pdf is given by

$$g_(x;\alpha,\beta,c) \;=\; e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right], \tag{1.5}$$

for $x > 0$, and $\alpha,\ \beta\ , c > 0$.

A motivation for developing this model is the following advantages:

- it posses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
- it provide better fits than other generalizations of the Weibull distributions in the literature;

- it is heavy tailed than the Weibull distributions hence can model different real data sets;
- it has a more flexible kurtosis than that of the Weibull distribution.

The rest of the paper is organized as follows: In section 2, some basic results, including the HLLLW distribution and its density function, series expansion, sub-models, hazard function and the quantile function are presented. The moments, moment generating function, mean and median deviations are given in section 3. Section 4 contains some additional useful results on the distribution of order statistics and Rényi entropy. In section 5, results on the estimation of the parameters of the HLLLW distribution via the method of maximum likelihood are presented. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators in section 6. Applications are given in section 7, followed by some concluding remarks.

2. HLLLW Distribution, Expansion of Density Function, Sub-models, Hazard and Quantile Functions

In this section, the HLLLW distribution, series expansion of its pdf, some sub-models, quantile function, hazard function as well some graphs are presented.

Substituting equation (1.4) into equation (1.2), we obtain the cdf of the half logistic log-logistic Weibull (HLLLW) distribution as

$$F(x;\alpha,\beta,c)=\frac{1-(1+x^{c})^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^{c})^{-1}e^{-\alpha x^{\beta}}}. \tag{2.1}$$

The corresponding pdf follows from inserting (1.4) and (1.5) into (1.3) and is given by

$$\begin{aligned} f(x;\alpha,\beta,c) &= 2e^{-\alpha x^{\beta}}(1+x^{c})^{-1}\left[(1+x^{c})^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right] \\ &\times \left[1+(1+x^{c})^{-1}e^{-\alpha x^{\beta}}\right]^{-2} \end{aligned} \tag{2.2}$$

for $\alpha,\beta,c>0$. If a random variable X has the half logistic log-logistic Weibull (HLLLW) density, we write $X\sim HLLLW(\alpha,\beta,c)$. Plots of the HLLLW pdf shows different shapes including right skewed, left skewed, decreasing, almost symmetric and reverse J-shapes.

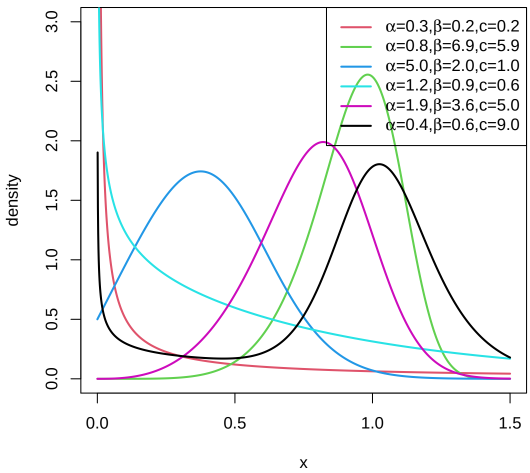

Figure 1: Plots of HLWLLoG Density Function

2.1. Expansion of Density Function

In this subsection, we present an expansion of the HLLLW density function. Note that

$$\begin{aligned} f(x;\alpha,\beta,c) &= \frac{2g_(x)}{\left[1+\overline{G}_(x)\right]^2} \\ &= 2\sum_{p=0}^{\infty}\frac{\Gamma(2+p)}{\Gamma(2)p!}(-1)^p\left[1-G_(x)\right]^p g_(x) \\ &= 2\sum_{p,k=0}^{\infty}\frac{\Gamma(2+p)}{\Gamma(2)p!}\binom{p}{k}(-1)^{p+k}\frac{k+1}{k+1}\left[G_(x)\right]^{k+1-1}g_(x) \\ &= \sum_{p,k=0}^{\infty}\frac{\Gamma(2+p)}{\Gamma(2)p!}\binom{p}{k}(-1)^{p+k}\frac{2}{k+1}g(x;\alpha,\beta,c,k+1), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} g(x;\alpha,\beta,c,k+1) &= \left(1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}\right)^{k+1-1} \\ &\times \ (k+1)e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right] \end{aligned}$$

is the exponentiated log-logistic Weibull (ELLW) density function with parameters $\alpha,\beta,c,k+1>0$. Thus, the HLLLW density function can be written as an infinite linear combination of the ELLW density functions. Consequently, the mathematical and statistical properties of the HLLLW distribution follows directly from those of the ELLW distribution.

2.2. Sub-models of HLLLW Distribution

In this subsection, we discuss some special models of the HLLLW distributions.

- If $\alpha \to 0$, the HLLLW model reduces to the Half logistic log-logistic model (HLLL) with pdf

$$f(x;c) = \frac{2(1+x^c)^{-2}cx^{c-1}}{\left[1+(1+x^c)^{-1}\right]^2}$$

for $c > 0$.

- If $\beta = 1$ we obtain the Half logistic log-logistic Exponential (HLLLE) with the pdf

$$f(x;\alpha,c) = \frac{2e^{-\alpha x}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+\alpha\right]}{\left[1+(1+x^c)^{-1}e^{-\alpha x}\right]^2}$$

for $\alpha, c > 0$.

- If $\beta = 2$, we obtain the Half logistic log-logistic Rayleigh (HLLLR) with the pdf

$$\begin{aligned} f(x;\alpha,c) &= 2e^{-\alpha x^2}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+2\alpha x\right] \\ &\times \left[1+(1+x^c)^{-1}e^{-\alpha x^2}\right]^{-2} \end{aligned}$$

for $\alpha, c > 0$.

- If $c = 1$, the HLLLW model reduces to a two parameter distribution with the pdf

$$\begin{aligned} f(x;\alpha,\beta) &= 2e^{-\alpha x^\beta}(1+x)^{-1}\left[(1+x)^{-1}+\alpha\beta x^{\beta-1}\right] \\ &\times \left[1+(1+x)^{-1}e^{-\alpha x^\beta}\right]^{-2} \end{aligned}$$

for $\alpha, \beta > 0$.

- If $c = \beta = 1$, the HLLLW model reduces to a one parameter distribution with the pdf,

$$\begin{aligned} f(x;\alpha) &= 2e^{-\alpha x}(1+x)^{-1}\left[(1+x)^{-1}+\alpha\right] \\ &\times \left[1+(1+x)^{-1}e^{-\alpha x}\right]^{-2} \end{aligned}$$

for $\alpha > 0$.

- If $\alpha = 1$, the HLLLW model reduces to a two parameter distribution with the pdf,

$$\begin{aligned} f(x;\beta,c) &= 2e^{-x^\beta}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+\beta x^{\beta-1}\right] \\ &\times \left[1+(1+x^c)^{-1}e^{-x^\beta}\right]^{-2} \end{aligned}$$

for $\beta, c > 0$.

- If $\alpha = c = 1$, the HLLLW model reduces to a one parameter distribution with the pdf,

$$\begin{aligned} f(x;\beta) &= 2e^{-x^\beta}(1+x)^{-1}\left[(1+x)^{-1}+\beta x^{\beta-1}\right] \\ &\times \left[1+(1+x)^{-1}e^{-x^\beta}\right]^{-2} \end{aligned}$$

for $\beta > 0$.

2.3. Hazard and Quantile Functions

In this section, we present the hazard and quantile functions of the HLLLW distribution. The hazard function is given by

$$\begin{aligned} h_F(x;\alpha,\beta,c) &= \frac{f_(x;\alpha,\beta,c)}{\overline{F}_(x;\alpha,\beta,c)} \\ &= \frac{2e^{-\alpha x^{\beta}}(1+x^c)^{-1}\left[(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right]}{\left(1-\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}\right)\left[1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}\right]^2}, \end{aligned} \tag{2.4}$$

for $\alpha,\beta,c>0$. Plots of the hazard function of the HLLLW distribution show different shapes including decreasing, increasing, bathtub followed by upside down, as well as bathtub shapes.

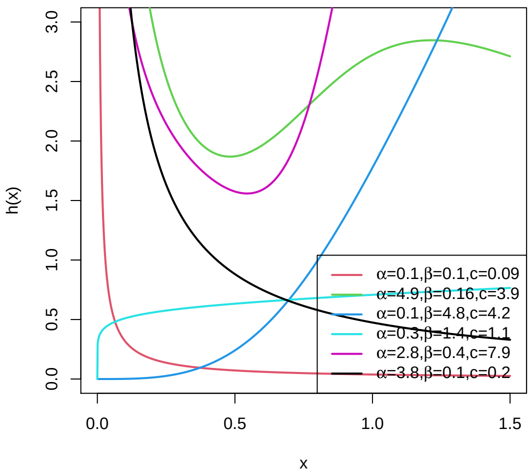

Figure 2: Plots of HLLLW Hazard Function

The quantile function of the HLLLW distribution is obtained by solving the non-linear equation:

$$F(x;\alpha,\beta,c)=\frac{1-(1+x^c)^{-1}e^{-\alpha x^{\beta}}}{1+(1+x^c)^{-1}e^{-\alpha x^{\beta}}}=u, \tag{2.5}$$

$0\le u\le 1$, that is,

$$\alpha x^{\beta}+\ln(1+x^c)+\ln\left(\frac{1-u}{1+u}\right)=0. \tag{2.6}$$

Therefore, random numbers can be generated from the HLLLW distribution by numerically solving the non-linear equation (2.6).

Table 1: **Table of Quantile for HLLLW Distribution**

	(α,β,c)				
u	$(0.1, 0.5, 4.02)$	$(0.2, 0.3, 3.8)$	$(0.02, 0.004, 2.1)$	$(0.3, 0.2, 1.4)$	$(0.2, 0.1, 0.9)$
0.1	0.7024	0.7091	0.4949	0.4154	0.2328
0.2	0.8641	0.8868	0.7289	0.7869	0.6025
0.3	0.9950	1.0333	0.9413	1.2242	1.1523
0.4	1.1185	1.1765	1.1649	1.8002	1.9866
0.5	1.2450	1.3252	1.4131	2.6226	3.3435
0.6	1.3898	1.5035	1.7190	3.9421	5.7227
0.7	1.5698	1.7270	2.1227	6.4557	10.5563
0.8	1.8300	2.0675	2.7542	13.1099	23.2730
0.9	2.3281	2.7459	4.0818	68.6328	85.6170

3. Moments, Conditional Moments and Mean Deviations

In this section, the r^{th} moment, conditional moments, mean deviations, Lorenz and Bonferroni curves of the HLLLW distribution are presented. Let $Y \sim ELLW(\alpha,\beta,c,k+1)$. Note that

$$\begin{aligned}
E(Y^r) &= \int_0^\infty y^r g_(y;\alpha,\beta,c,k+1)dy \\
&= \int_0^\infty y^r (k+1)\left(1-(1+y^c)^{-1}e^{-\alpha y^\beta}\right)^{(k+1)-1} e^{-\alpha y^\beta}(1+y^c)^{-1} \\
&\times \left[(1+y^c)^{-1}cy^{c-1}+\alpha\beta y^{\beta-1}\right]dy \\
&= \sum_{i=0}^\infty \binom{(k+1)-1}{i}(-1)^i(k+1)\int_0^\infty y^r(1+y^c)^{-i-1}e^{-\alpha y^\beta(i+1)} \\
&\times \left[(1+y^c)^{-1}cy^{c-1}+\alpha\beta y^{\beta-1}\right] \\
&= \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i}\left[c\int_0^\infty y^{r+\beta j+c-1}\right. \\
&\times \left.(1+y^c)^{-i-2}dy+\alpha\beta\int_0^\infty y^{r+\beta(j+1)-1}(1+y^c)^{-i-1}dy\right].
\end{aligned}$$

We note that by applying the substitution $t=(1+y^c)^{-1}$, we have

$$\begin{aligned}
E(Y^r) &= \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i} \\
&\times \left[\int_0^1 t^{i-\frac{r+\beta j+c}{c}-1}(1-t)^{\frac{r+\beta j+c}{c}-1}dt\right. \\
&+ \left.\alpha\beta\int_0^1 t^{i-1-\frac{r+\beta(j+1)}{c}-1}(1-t)^{\frac{r+\beta(j+1)}{c}-1}dt\right].
\end{aligned}$$

Consequently,

$$E(Y^r) = \sum_{i,j=0}^\infty \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i}$$

$$\times \quad \left[B\left(i-\frac{r+\beta j+c}{c},\frac{r+\beta j+c}{c}\right)\right.$$
$$+ \quad \left. B\left(i-1-\frac{r+\beta(j+1)}{c},\frac{r+\beta(j+1)}{c}\right)\right]. \tag{3.1}$$

Consequently, the r^{th} moment of the HLLLW distribution is given by

$$E(X^r)=\sum_{p,k=0}^{\infty}\frac{\Gamma(2+p)}{\Gamma(2)p!}\binom{p}{k}(-1)^{p+k}\frac{2}{k+1}E(Y^r), \tag{3.2}$$

where $E(Y^r)$ is given by equation (3.1). The moment generating function of the ELLW distribution is given by $E(e^{tY})=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(Y^r)$, where $E(Y^r)$ is given by the equation (3.1). The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), CV, CS and CK are given by

$$\sigma^2=\mu_2'-\mu^2, \quad CV=\frac{\sigma}{\mu}=\frac{\sqrt{\mu_2'-\mu^2}}{\mu}=\sqrt{\frac{\mu_2'}{\mu^2}-1},$$

$$CS=\frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}}=\frac{\mu_3'-3\mu\mu_2'+2\mu^3}{(\mu_2'-\mu^2)^{3/2}},$$

and

$$CK=\frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2}=\frac{\mu_4'-4\mu\mu_3'+6\mu^2\mu_2'-3\mu^4}{(\mu_2'-\mu^2)^2},$$

respectively. Some moments for selected parameters values are given in Table 2 and plots are presented in Figure 4. Plots of skewness and kurtosis for choices of the model parameters reveal that skewness and kurtosis depend on the shape parameters β and c.

Table 2: Table of Moments for Selected Parameters for HLLLW Distribution

	(α,β,c)				
	(0.04,0.8,0.2)	(0.2,1.5,0.7)	(0.9,0.2,0.5)	(0.6,0.9,1.0)	(0.9,0.8,0.4)
E(X)	0.0146	0.0634	0.0604	0.1181	0.0890
E(X^2)	0.0073	0.0351	0.0292	0.0659	0.0502
E(X^3)	0.0049	0.0241	0.0190	0.0448	0.0349
E(X^4)	0.0036	0.0183	0.0140	0.0336	0.0267
E(X^5)	0.0029	0.0148	0.0111	0.0268	0.0216
E(X^6)	0.0024	0.0124	0.0092	0.0223	0.0182
SD	0.0021	0.0106	0.0078	0.0190	0.0157
CV	0.0018	0.0093	0.0068	0.0166	0.0138
CS	0.0016	0.0083	0.0060	0.0147	0.0123
CK	0.0014	0.0075	0.0054	0.0132	0.0111

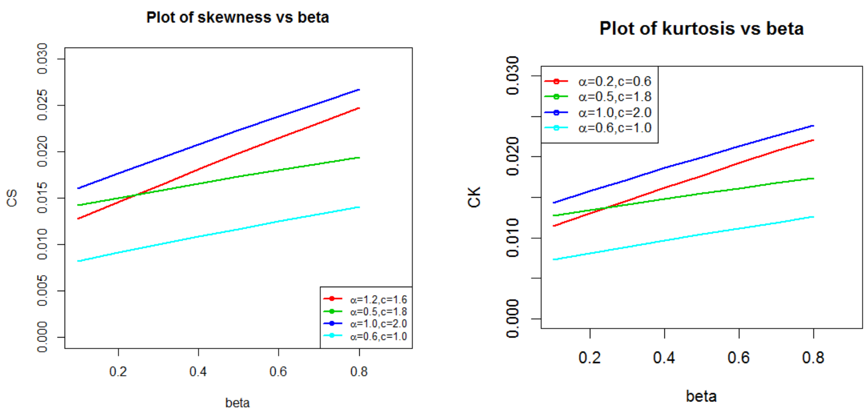

Figure 3: Plots of Skewness and Kurtosis for shape parameters β

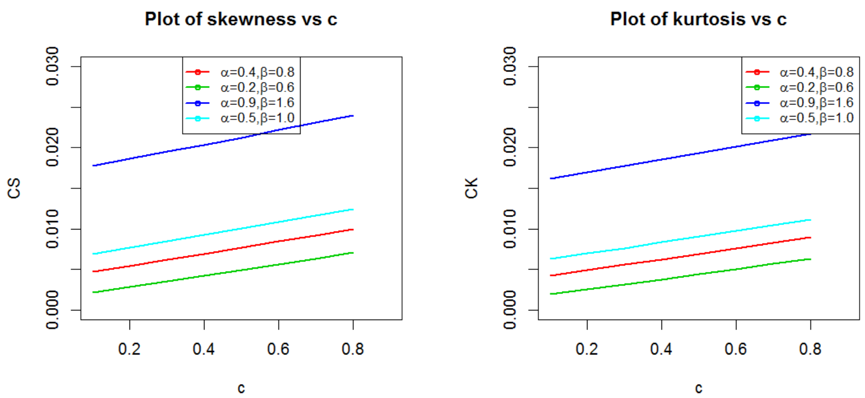

Figure 4: Plots of Skewness and Kurtosis for shape parameters c

3.1. Conditional Moments

For lifetime models, it is also of interest to obtain the r^{th} conditional moments and the mean residual lifetime function. The r^{th} conditional moment for the HLLLW distribution can be readily obtained from those of the ELLW distribution. The conditional r^{th} moment for the ELLW distribution is given by

$$\begin{aligned} E(Y^r|Y>t) &= \frac{1}{\overline{G}_(t)}\int_t^{\infty} x^r g_(y;\alpha,\beta,c,k+1)dy \\ &= \frac{1}{\overline{G}_(t)}\sum_{i,j=0}^{\infty}\frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i} \\ &\times \left[B_{(1+t^c)^{-1}}\left(i-\frac{r+\beta j+c}{c},\frac{r+\beta j+c}{c}\right)\right. \end{aligned}$$

$$+ \quad B_{(1+t^c)^{-1}}\left(i-1-\frac{r+\beta(j+1)}{c}, \frac{r+\beta(j+1)}{c}\right)\Bigg], \tag{3.3}$$

where $B_{(1+t^c)^{-1}}(a,b)$ is the incomplete beta function. Consequently, the r^{th} conditional moment of the HLLLW distribution is given by

$$\begin{aligned}
E(X^r|X>t) &= \sum_{p,k=0}^{\infty} \frac{\Gamma(2+p)}{\Gamma(2)p!}\binom{p}{k}(-1)^{p+k}\frac{2}{k+1}E(Y^r|Y>t) \\
&= \frac{1}{\overline{G}_(t)} \sum_{p,k=0}^{\infty} \frac{\Gamma(2+p)}{\Gamma(2)p!}\binom{p}{k}(-1)^{p+k}\frac{2}{k+1} \\
&\times \sum_{i,j=0}^{\infty} \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i} \\
&\times \Bigg[B_{(1+t^c)^{-1}}\left(i-\frac{r+\beta j+c}{c}, \frac{r+\beta j+c}{c}\right) \\
&+ B_{(1+t^c)^{-1}}\left(i-1-\frac{r+\beta(j+1)}{c}, \frac{r+\beta(j+1)}{c}\right)\Bigg].
\end{aligned} \tag{3.4}$$

3.2. Mean Deviation, Lorenz and Bonferroni Curves

In this subsection, we present Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the HLLLW distribution. Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality.

3.3. Mean Deviations

If Y has the ELLW distribution, we can derive the mean deviation about the mean μ by

$$\delta_1 = \int_0^{\infty} |y-\mu| f_(y)dy = 2\mu F_Y(\mu) - 2\mu + 2T(\mu),$$

and the median deviation about the median M by

$$\delta_2 = \int_0^{\infty} |y-M| f_Y(y)dy = 2T(M) - \mu,$$

where $\mu = E(Y)$ is given in equation (3.1) with $r = 1$, M the median of $F_Y(x)$ and $T(a) = \int_a^{\infty} x \cdot f_Y(y)dy$. Note that

$$\begin{aligned}
T(a) &= \sum_{i,j=0}^{\infty} \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i} \\
&\times \Bigg[B_{(1+a^c)^{-1}}\left(i-\frac{1+\beta j+c}{c}, \frac{1+\beta j+c}{c}\right) \\
&+ B_{(1+a^c)^{-1}}\left(i-1-\frac{1+\beta(j+1)}{c}, \frac{1+\beta(j+1)}{c}\right)\Bigg].
\end{aligned} \tag{3.5}$$

Consequently, the mean deviations for HLLLW can be readily obtained from those of the ELLW distribution.

3.4. Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves for the ELLW are given as

$$B(p) = \frac{1}{p\mu}\int_0^q y f_{_Y}(y)dy = \frac{1}{p\mu}[\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu}\int_0^q y f_{_Y}(y)dy = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty y f_{_Y}(y)dy$, $q = F_{_Y}^{-1}(p)$, $0 \le p \le 1$. It follows therefore that Bonferroni and Lorenz curves for the HLLLW distribution can be readily obtained from those of the ELLW distribution.Note that

$$\begin{aligned} T(q) &= \sum_{i,j=0}^{\infty} \frac{(k+1)(-1)^{i+j}[\alpha(i+1)]^j}{j!}\binom{(k+1)-1}{i} \\ &\times \Bigg[B_{(1+q^c)^{-1}}\left(i - \frac{1+\beta j + c}{c}, \frac{1+\beta j + c}{c}\right) \\ &+ B_{(1+q^c)^{-1}}\left(i - 1 - \frac{1+\beta(j+1)}{c}, \frac{1+\beta(j+1)}{c}\right)\Bigg]. \end{aligned} \tag{3.6}$$

Consequently, the mean deviations for HLLLW distribution can be readily obtained from those of the ELLW distribution.

4. Order Statistics and Rényi Entropy

The concept of entropy plays a very important role in information theory. In this section, we present the distribution of the order statistic and Rényi entropy for the HLLLW distribution.

4.1. Distribution of Order Statistics

Order statistics play an important role in probability and statistics. Le $X_1, X_2, ..., X_n$ be a random sample from the HLLLW distribution and suppose $X_{1:n} < X_{2:n}, ... < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l f(x)\left[F(x)\right]^{k+l-1}. \tag{4.1}$$

Note, $f(x)\left[F(x)\right]^{k+l-1} = \frac{1}{k+l}\frac{d}{dx}\left[F(x)\right]^{k+l}$. The corresponding pdf of $f_{k:n}(x)$ is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}\frac{d}{dx}\left[F(x)\right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}\frac{d}{dx}\left[\frac{1-(1+x^c)^{-1}e^{-\alpha x^\beta}}{1+(1+x^c)^{-1}e^{-\alpha x^\beta}}\right]^{k+l} \\ &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}\frac{d}{dx}\left(G_(x;\lambda,c,\gamma,k+l)\right), \end{aligned}$$

where $G_(x;\alpha,\beta,c,k+l)$ is exponentiated half logistic log-logistic Weibull (EHLLLW) distribution function with parameters α,β,c and $k+l>0$. Thus, the pdf of the k^{th} order statistic can be expressed as a linear combination of the pdf of the EHLLLW distribution.

4.2. Rényi Entropy

In this section, we present Rényi entropy for the HLLLW distribution. Rényi entropy (Rényi [28]) is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty [f(x;\alpha,\beta,c)]^v dx\right), v\neq 1, v>0. \tag{4.2}$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Note that

$$\begin{aligned}
f^v(x) &= 2^v e^{-\alpha v x^\beta}(1+x^c)^{-v}\left[(1+x^c)^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right]^v\\
&\times \left[1+(1+x^c)^{-1}e^{-\alpha x^\beta}\right]^{-2v}\\
&= \sum_{p=0}^\infty 2^v e^{-\alpha v x^\beta}(1+x^c)^{-v}\binom{v}{p}(1+x^c)^{-p}c^p x^{pc-p}x^{(\beta-1)(v-p)}\\
&\times (\alpha\beta)^{v-p}\sum_{z=0}^\infty\frac{\Gamma(2v+z)}{\Gamma(2v)z!}(-1)^z(1+x^c)^{-z}e^{-\alpha z x^\beta}\\
&= \sum_{p,z=0}^\infty 2^v(\alpha\beta)^{v-p}c^p\binom{v}{p}\frac{\Gamma(2v+z)}{\Gamma(2v)z!}(-1)^z(1+x^c)^{-z-p-v}\\
&\times x^{pc-p+(\beta-1)(v-p)}e^{-\alpha x^\beta(z+v)}\\
&= \sum_{p,z,s=0}^\infty 2^v(\alpha\beta)^{v-p}c^p\left[\alpha(z+v)\right]^s\binom{v}{p}\frac{\Gamma(2v+z)}{\Gamma(2v)z!}(-1)^{z+s}\\
&\times x^{pc+\beta s-p+(\beta-1)(v-p)}(1+x^c)^{-z-p-v}.
\end{aligned}$$

Note that, by applying the substitution $y=(1+x^c)^{-1}$, we have the following

$$\begin{aligned}
\int_0^\infty \frac{x^{pc+\beta s-p+(\beta-1)(v-p)}}{(1+x^c)^{z+p+v}}dx &= \frac{1}{c}B\Big(z+p+v-2-\frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\\
&, \quad \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\Big).
\end{aligned}$$

Consequently, Rényi entropy for the HLLLW distribution reduces to

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\left(\sum_{p,z,s=0}^\infty 2^v(\alpha\beta)^{v-p}c^p\left[\alpha(z+v)\right]^s\binom{v}{p}\frac{\Gamma(2v+z)}{\Gamma(2v)z!}(-1)^{z+s}\right.\\
&\times \frac{1}{c}B\Big(z+p+v-2-\frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\\
&, \left.\quad \frac{pc+\beta s-p+(\beta-1)(v-p)+1}{c}\Big)\right),
\end{aligned}$$

for $v\neq 1$ and $v>0$.

5. Estimation

Let $X_i \sim HLLLW(\alpha,\beta,c)$ and $\Delta = (\alpha,\beta,c)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned}\ell &= n\ln(2) - \alpha\sum_{i=1}^{n} x_i^{\beta} - \sum_{i=1}^{n}\ln(1+x_i^c) + \sum_{i=1}^{n}\ln\left[(1+x_i^c)^{-1}cx_i^{c-1} + \alpha\beta x_i^{\beta-1}\right] \\ &- 2\sum_{i=1}^{n}\ln\left[1+(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta}}\right]. \qquad (5.1)\end{aligned}$$

Elements of the score vector $U = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial c})$ can be readily obtained. The equations obtained by setting the partial derivatives to zero are not in closed form and the values of the parameters α, c, β must be found via iterative methods. The maximum likelihood estimates (MLE) of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial c})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{3X3} = E(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j})$, $i,j = 1,2,3$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\left.\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}}\right]_{3X3}, \quad i,j = 1,2,3. \qquad (5.2)$$

For a given set of observations, the matrix given in equation (5.2) is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software.
The multivariate normal distribution $N_3(\underline{0}, J^{-1}(\hat{\mathbf{\Delta}}))$, where the mean vector $\underline{\mathbf{0}} = (0,0,0)^T$ and $J^{-1}(\hat{\mathbf{\Delta}})$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for λ, c, and δ are given by:

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{c} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \text{and} \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})},$$

respectively, where $\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})$, and $\mathbf{I}_{\delta\delta}^{-1}(\widehat{\mathbf{\Delta}})$, are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}}) = (n\mathbf{I}(\hat{\mathbf{\Delta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

6. Monte Carlo Simulations

In this section, the performance of the maximum likelihood estimates is examined by conducting simulation studies for different sample sizes. We examine the performance of the HLLLW distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800, 1200) via the R package. We simulate $N = 1000$ samples for the true parameters values given in the Table 3 and Table 4. The Average Bias and Root Mean Square Error (RMSE) were computed. The average bias and RMSE for the estimated parameter $\hat{\theta}$, say, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The table lists the mean MLEs of the parameters along with the respective root mean squared errors (RMSEs).

Table 3: **Monte Carlo Simulation Results**

		(0.4,0.9,0.8)			(1.4,1.5,1.8)			(2.0,1.8,0.5)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.3770	0.2238	-0.0229	1.4484	0.4283	0.0484	2.2140	0.8278	0.2140
	50	0.3778	0.1647	-0.0221	1.4140	0.2665	0.0140	2.0932	0.3781	0.0932
	100	0.3889	0.1113	-0.0110	1.4095	0.1932	0.0095	2.0461	0.2593	0.0461
	200	0.3952	0.0788	-0.0047	1.3945	0.1301	-0.0054	2.0124	0.1662	0.0124
	400	0.3982	0.0557	-0.0017	1.3959	0.0918	-0.0040	2.0036	0.1182	0.0036
	800	0.3986	0.0411	-0.0013	1.3963	0.0640	-0.0036	1.9981	0.0872	-0.0018
	1200	0.3992	0.0307	-0.0007	1.3969	0.0505	-0.0030	2.0004	0.0666	0.0004
β	25	1.1515	0.6706	0.2515	1.7665	0.6636	0.2665	1.9857	0.7112	0.1857
	50	1.0041	0.3799	0.1041	1.6411	0.4169	0.141	1.8706	0.3721	0.0706
	100	0.9392	0.2029	0.0392	1.5879	0.2907	0.0879	1.8417	0.2574	0.0417
	200	0.9114	0.1297	0.0114	1.5541	0.2105	0.0541	1.8204	0.1685	0.0204
	400	0.9042	0.0905	0.0042	1.5327	0.1615	0.0327	1.8136	0.1149	0.0136
	800	0.9033	0.0613	0.0033	1.5205	0.1313	0.0205	1.8052	0.0777	0.0052
	1200	0.9011	0.0482	0.0011	1.5150	0.1145	0.0150	1.8050	0.0666	0.0050
c	25	0.8826	0.3218	0.0826	1.9035	0.9145	0.1035	0.6457	0.3520	0.1457
	50	0.8632	0.2515	0.0632	1.9477	0.9020	0.1477	0.5795	0.2586	0.0795
	100	0.8390	0.1721	0.0390	1.8668	0.6362	0.0668	0.54749	0.1730	0.0474
	200	0.8304	0.1384	0.0304	1.8438	0.4899	0.0438	0.5265	0.0688	0.0265
	400	0.8158	0.0857	0.0158	1.8388	0.4158	0.0388	0.5232	0.0495	0.0232
	800	0.8079	0.0554	0.0079	1.8248	0.3487	0.0248	0.5206	0.0365	0.0206
	1200	0.8052	0.0445	0.0052	1.8231	0.3025	0.0231	0.5197	0.0312	0.0197

Table 4: **Monte Carlo Simulation Results**

		(2.0,2.8,2.0)			(3.0,2.5,2.5)			(3.0,3.2,3.4)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	2.2207	0.7338	0.2207	3.4622	1.2786	0.4622	3.4705	1.3094	0.4705
	50	2.0876	0.3748	0.0876	3.2513	0.7018	0.2513	3.2182	0.7654	0.2182
	100	2.0431	0.2376	0.0431	3.1385	0.4579	0.1385	3.1235	0.4219	0.1235
	200	2.0198	0.1620	0.0198	3.0656	0.2854	0.0656	3.0672	0.2715	0.0672
	400	2.0141	0.1205	0.0141	3.0332	0.1829	0.0332	3.0343	0.1835	0.0343
	800	2.0087	0.0849	0.0087	3.0290	0.1321	0.0290	3.0314	0.1332	0.0314
	1200	2.0022	0.0649	0.0022	3.0185	0.1086	0.0185	3.0152	0.1037	0.0152
β	25	3.2089	1.0516	0.4089	2.9459	0.9958	0.4459	3.7442	1.1905	0.5442
	50	2.9372	0.5824	0.1372	2.7485	0.5654	0.2485	3.4776	0.7698	0.2776
	100	2.8377	0.4105	0.0377	2.6677	0.4040	0.1677	3.3758	0.5001	0.1758
	200	2.7816	0.3171	-0.0183	2.6009	0.2851	0.1009	3.3013	0.3739	0.1013
	400	2.7870	0.2418	-0.0129	2.5612	0.2069	0.0612	3.2674	0.2756	0.0674
	800	2.7904	0.1766	-0.0095	2.5454	0.1665	0.0454	3.2526	0.2110	0.0526
	1200	2.7916	0.1449	-0.0083	2.5345	0.1433	0.0345	3.2431	0.2005	0.0431
c	25	2.2358	0.8174	0.2358	3.3568	17.4269	0.8568	4.0698	5.9043	0.6698
	50	2.2474	0.7931	0.2474	2.6406	3.3831	0.1406	4.0007	6.2780	0.6007
	100	2.2319	0.7259	0.2319	2.4733	0.7425	-0.0266	3.6606	2.3400	0.2606
	200	2.2086	0.6263	0.2086	2.4779	0.6413	-0.0220	3.5826	1.6641	0.1820
	400	2.1423	0.4936	0.1423	2.4783	0.5246	-0.0216	3.4765	0.9394	0.0765
	800	2.0935	0.3592	0.0935	2.5126	0.4851	0.0126	3.4605	0.7565	0.0605
	1200	2.0588	0.2875	0.0588	2.5002	0.4322	0.0002	3.4442	0.7111	0.0442

From Table 3 and Table 4 above we conclude that estimation method is adequate as the simulated estimates are closed to the true values of parameters. We also observed that estimated root mean square errors (RMSE_s) consistently decreases with increasing sample size and the average bias decreases as the sample size n increases.

7. Applications

The flexibility and usefulness of the HLLLW distribution and its sub-models for data modeling is illustrated in this section. We compare the HLLLW distribution with the genereralized Weibull distribution (Dimitrakopoulou et al. [14]), Lindley-Weibull (LW) distribution (Asgharzadeh et al. [6]), Log-Logistic Weibull (LLW) distribution (Oluyede et al. [26])), the half logistic genereralized Weibull (HLGW) distribution (Anwar and Bibi [3]), exponentiated modified Weibull distribution (Elbatal [15]) and the Topp-Leone generated Weibull (TLGW) distribution (Aryal et al. [5]). The pdf of the generalized Weibull (GW) distibution is given by

$$g_(x;w,\lambda,\gamma)=w\lambda\gamma x^{\lambda-1}\left(1+\gamma x^{\lambda}\right)^{w-1}\exp\left(1-(1+\gamma x^{\lambda})^{w}\right),$$

for $w,\lambda,\gamma>0$ and $x>0$.

The LW distribution (see Asgharzadeh et al. [6] for details) has the pdf given by

$$g_(x;\lambda,\alpha,\beta)=\frac{e^{-\lambda x-\alpha x^{\beta}}}{1+\lambda}[\lambda^2(1+x)+(1+\lambda+\lambda x)\alpha\beta x^{\beta-1}],$$

for $\lambda,\alpha,\beta>0$ and $x>0$.The pdf of the Log-logistic Weibull (LLW) distribution is given by

$$g_(x;c,\alpha,\beta)=(1+x^{c})^{-1}e^{-\alpha x^{\beta}}\left[(1+x^{c})^{-1}cx^{c-1}+\alpha\beta x^{\beta-1}\right],$$

for $c,\alpha,\beta>0$, and $x\geq 0$. The pdf of the half logistic genereralized Weibull (HLGW) distribution is given by

$$g_(x;w,\lambda,\gamma)=\frac{2w\lambda\gamma x^{\lambda-1}\left(1+\gamma x^{\lambda}\right)^{w-1}\exp\left(1-(1+\gamma x^{\lambda})^{w}\right)}{\left[1+\exp(1-(1+\gamma x^{\lambda})^{w})\right]^2},$$

for $w,\lambda,\gamma>0$, and $x\geq 0$. The pdf of exponentiated modified Weibull (EMW) is given by

$$g_(x;\gamma,\delta,\lambda,\theta)=\gamma\left[\delta+\lambda\theta^{\lambda}x^{\lambda-1}\right]\exp(-(\delta x+(\theta x)^{\lambda}))\left[1-\exp(-(\delta x+(\theta x)^{\lambda}))\right]^{\gamma-1},$$

for $\gamma,\delta,\lambda,\theta>0$, and $x\geq 0$. The pdf of Topp-Leone generated Weibull (TLGW) distribution is given by

$$\begin{aligned}g_(x;\alpha,\theta,\lambda,\beta) &= 2\alpha\theta\beta\lambda^{\beta}x^{\beta-1}\exp\left(-(\lambda x)^{\beta}\right)\left[(1-\exp(-(\lambda x)^{\beta})\right]^{\theta\alpha-1}\\ &\times (1-(1-\exp(-(\lambda x)^{\beta}))^{\theta})(2-(1-\exp(-(\lambda x)^{\beta}))^{\theta})^{\alpha-1},\end{aligned}$$

for $\alpha,\theta,\lambda,\beta>0$, and $x\geq 0$.

The maximum likelihood estimates (MLEs) of the HLLLW parameters and its sub-models are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC=2p-2\ln(L)$), Bayesian Information Criterion ($BIC=p\ln(n)-2\ln(L)$), and Consistent Akaike Information Criterion ($AICC=AIC+2\frac{p(p+1)}{n-p-1}$), where $L=L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are used to assess the performance of the models. Tables 6 and 8 shows results for the data set for HLLLW distribution, its sub-models and several non-nested models.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [9]) are given in Figure 5 and Figure 6. For the probability plot, we plotted $G(x_{(j)};\hat{\alpha},\hat{c},\hat{\beta})$ against $\dfrac{j-0.375}{n+0.25}$, $j=1,\,2,\,\cdots,\,n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS=\sum_{j=1}^{n}\left[G(x_{(j)};\hat{\alpha},\hat{c},\hat{\beta})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [10] are presented in the tables. The Kolmogorov-Smirnov (KS) and its P-value are also presented in Tables 6 and 8. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of KS, W^* and A^*, the better the fit.

7.1. Electronics Data

The data for lifetimes of 20 electronic components obtained from Murthy et al. [23] are given as
0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79,1.80,1.94,2.38,2.40,2.87,2.99, 3.14,3.17,4.72,5.09. Estimates of the parameters of HLLLW distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W^*, A^*, Kolmogorov-Smirnov (KS) and its P-value as well as SS are given in Table 6. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 5.

Table 5: Estimates of Models for Lifetimes of 20 Electronic Components Data

	Estimates			
Model	α	β	c	
HLLLW	0.0613	2.4057	0.8223	
	(0.0827)	(0.8385)	(0.3242)	
HLLL	0	0	1.5334	
	-	-	(0.2692)	
HLLLR	0.3056	2	0.9868	
	(0.1736)	-	(0.4452)	
HLLLW(1, β, c)	1	0.7358	0.5908	
	-	(0.1740)	(0.3861)	
	W	λ	γ	
GW	0.5390	1.2807	0.8869	
	(0.1375)	(0.2582)	(0.3712)	
	λ	α	β	
LW	0.7740	0.0346	0.5474	
	(0.2021)	(0.1779)	(0.9914)	
	c	α	β	
LLW	0.9210	0.0145	3.1704	
	(0.2984)	(0.0287)	(1.2036)	
	w	λ	γ	
HLGW	8.7112×10^{02}	2.5024×10^{-01}	1.0030×10^{-03}	
	(3.7763×10^{-09})	(5.3429×10^{-02})	(1.1285×10^{-04})	
	γ	δ	λ	θ
EMW	1.1392	5.5964×10^{-01}	1.7001×10^{01}	1.0000×10^{-04}
	(3.3249×10^{-01})	(1.5419×10^{-01})	(1.3217×10^{-18})	(2.2485×10^{-13})
	α	θ	λ	β
TLGW	0.3572	0.5187	0.1881	4.3262
	(0.5418)	(0.9783)	(0.0484)	(4.5895)

Table 6: Goodness of fit-statistics for Lifetimes of 20 Electronic Components Data

	Statistics								
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
HLLLW	62.9432	68.9432	70.44325	71.93044	0.0288	0.2022	0.1194	0.9058	0.0368
HLLL	71.9342	73.9342	74.1564	74.9299	0.1572	0.9015	0.1542	0.6723	0.0865
HLLLR	68.1241	72.1241	72.8300	74.11559	0.0997	0.5790	0.2339	0.1907	0.2397
HLLLW(1, β, c)	79.5829	83.5829	84.2887	85.5743	0.0904	0.5281	0.4486	0.0003	1.3743
GW	68.6887	74.6888	76.1888	77.6760	0.1154	0.6664	0.1587	0.6384	0.0696
LW	65.0536	71.0536	72.5536	74.0408	0.0610	0.3644	0.1352	0.8108	0.0628
LLW	66.3973	72.3973	73.8973	75.3845	0.0539	0.3194	0.2642	0.101	0.3366
HLGW	82.81607	88.82693	90.32693	91.81413	0.0913	0.5322	0.3518	0.0101	0.8084
EMW	66.22044	74.22044	76.88711	78.20337	0.0845	0.4934	0.1573	0.6490	0.0911
TLGW	62.9637	71.7721	74.43876	75.75503	0.0494	0.2936	0.1336	0.8219	4.3075

The values in table 6 show that the HLLLW distribution gives the smallest values for the Goodness-of-Fit statistics and the greatest p-value of KS-test. Thus, the HLLLW distribution provides better fit than the rest of the distributions for the lifetimes of 20 electronic components data. Additionaly, the value of sum of squares (SS=0.0368) from the probability plots in Figure 5 is smaller for HLLLW distribution comfirming that the HLLLW distribution is more appropriate to represent the lifetimes of 20 electronic components data than the rest distributions.

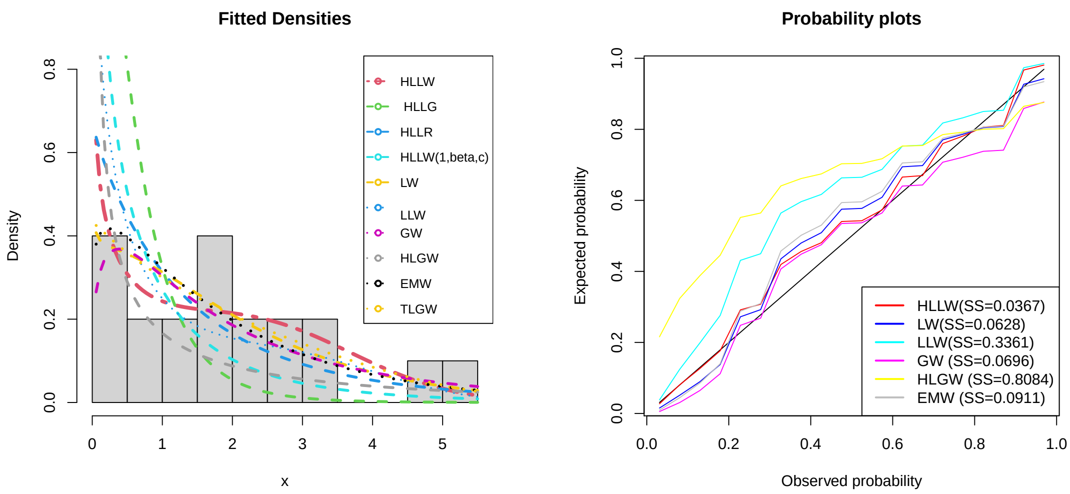

Figure 5: Fitted Densities and Probability Plots of Lifetimes of 20 Electronic Components Data

7.2. Time to failure of kevlar 49/epoxy strands tested at various stress level data

The data consists of 101 data points representing the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 90% stress level until all had failed, so that we have complete data with exact times of failure. The failure times in hours from Barlow et al. [7] are shown below
0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.04,0.05,0.06,0.07,0.07,0.08,0.09,0.09,
0.10,0.10,0.11,0.11,0.12,0.13,0.18,0.19,0.20,0.23,0.24,0.24,0.29,0.34,0.35,
0.36,0.38,0.40,0.42,0.43,0.52,0.54,0.56,0.60,0.60,0.63,0.65,0.67,0.68,0.72,
0.72,0.72,0.73,0.79,0.79,0.80,0.80,0.83,0.85,0.90,0.92,0.95,0.99,1.00,1.01,

1.02,1.03,1.05,1.10,1.10,1.11,1.15,1.18,1.20,1.29,1.31,1.33,1.34,1.40,1.43,
1.45,1.50,1.51,1.52,1.53,1.54,1.54,1.55,1.58,1.60,1.63,1.64,1.80,1.80,1.81,
2.02,2.05,2.14,2.17,2.33,3.03,3.03,3.34,4.20,4.69,7.89.
Estimates of the parameters of HLLLW distribution and its related sub-models (standard error in parentheses), AIC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (KS) and its P-value as well as SS are given in Table 8. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.

Table 7: MLEs of the parameters, SEs in parenthesis for keylar 49/epoxy failure time data

	Estimates			
Model	α	β	c	
HLLLW	0.7797	0.5256	2.2184	
	(0.1249)	(0.0941)	(0.4272)	
HLLL	0	0	1.2229	
	-	(-)	(0.1086)	
HLLLR	0.2311	2	0.7482	
	0.0435	(-)	(0.1053)	
HLLLW(1, β, c)	1	0.8914	0.7852	
	-	(0.0944)	-	
	W	λ	γ	
GW	0.9898	0.9263	1.0357	
	(0.4968)	(0.1507)	(0.7832)	
	λ	α	β	
LW	0.8890	0.4716	0.7481	
	(0.5527)	(0.4809)	(0.2543)	
	c	α	β	
LLW	2.1997	0.4113	0.5412	
	(0.3761)	(0.0919)	(0.1021)	
	w	λ	γ	
HLGW	1.1011	0.7790	1.2649	
	(0.5262)	(0.1419)	(0.9219)	
	γ	δ	λ	θ
EMW	8.6634×10^{-01}	8.8828×10^{-01}	1.7894×10^{01}	1.0000×10^{-04}
	(1.0983×10^{-01})	(1.2011×10^{-01})	(1.5316×10^{-19})	(2.7403×10^{-14})
	α	θ	λ	β
TLGW	0.3191	3.0963	0.6765	0.8652
	(0.4007)	(4.8575)	(0.4922)	(0.2998)

Table 8: Goodness-of-fit statistics for kevlar 49/epoxy failure time data

	Statistics								
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	P-value	SS
HLLLW	203.5713	209.5713	209.8187	217.4166	0.0822	0.5948	0.0700	0.7049	0.1038
HLLL	282.8573	284.8573	284.8977	287.4725	0.3966	2.1424	0.3834	2.5410×10^{-13}	5.7580
HLLLR	233.7703	237.7703	237.8928	243.0006	0.2067	1.5106	0.2211	0.0001	1.6510
HLLLW(1, β, c)	207.1673	211.1673	211.2898	216.3976	0.1718	0.9811	0.1422	0.0335	0.5198
GW	205.9939	211.9939	212.2413	219.8392	0.2000	1.1177	0.0948	0.3236	0.2093
LW	205.2548	211.2548	211.5023	219.1002	0.1479	0.8797	0.0785	0.5617	0.1441
LLW	207.4965	213.4965	213.7439	221.3419	0.1343	0.8644	0.0998	0.2664	0.2699
HLGW	204.1581	210.1581	210.4055	218.0034	0.1362	0.8161	0.0733	0.6488	0.1325
EMW	205.6399	213.6399	214.0566	224.1004	0.1786	1.0183	0.0887	0.4045	0.1795
TLGW	205.2654	213.2654	213.6821	223.7259	0.1625	0.9428	0.0845	0.4661	0.6362

The values of the goodness-of-fit statistics: W^*, A^*, KS and its p-value clearly show that the HLLLW distribution is by far the better fit for the kevlar 49/epoxy failure time data. The value of sum of squares (SS=0.1038) from the probability plots in Figure 6 is smaller for HLLLW distribution. Also, the goodness-of-fit statistics KS, W^* and A^* are smaller for HLLLW distribution as compared to nested and non-nested distributions for the kevlar 49/epoxy failure time data, hence we can conclude that the HLLLW distribution is by far better fit.

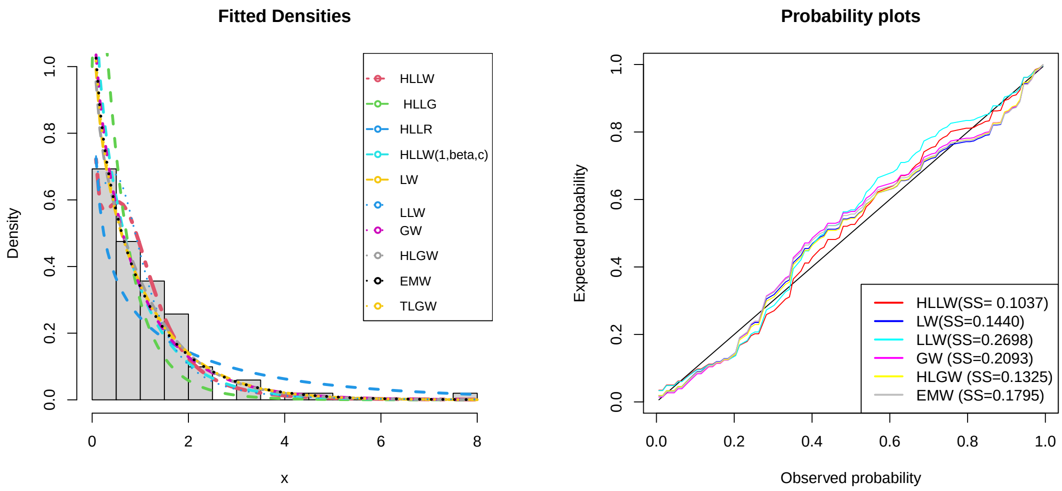

Figure 6: Fitted Densities and Probability Plots for Time to Failure of kevlar 49/epoxy strands tested at various stress level data

7.3. Likelihood Ratio Test

The likelihood ratio (LR) test can be used to compare the fit of the HLLLW distribution with its sub-models for a given data set. For example, to test $\beta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{\alpha}, \hat{c}, \hat{\beta})) - \ln(L(\tilde{\alpha}, \tilde{c}, 1))]$, where $\hat{\alpha}$, $\hat{c}$, and $\hat{\beta}$ are the unrestricted estimates, and $\tilde{\alpha}$, and $\tilde{c}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom. The likelihood ratio test results for comparing the full and nested models are given below.

Table 9: Likelihood ratio test results

		Keylar Eproxy Data	Electronics Data
Model	df	$\chi^2(p-value)$	$\chi^2(p-value)$
HLLL	2	78.2860 (<0.00001)	8.991 (0.0111)
HLLLR	1	30.1990 (<0.00001)	5.1809 (0.0143)
HLLLW(1, β, c)	1	3.596(0.0579)	16.6397 (0.00004)

From Table 9, we conclude that there are significant differences between HLLL and HLLLW distributions, HLLLR and HLLLW distributions as well as between $HLLLoG(1, \beta, c)$ and HLLLW distributions at the 10% level of significance based on the LR tests among all the datasets.

8. Concluding Remarks

We have proposed and studied a new three parameter distribution called the Half Logistic Log-Logistic Weibull (HLLLW) distribution. Distributional properties of this model are derived. Maximum likelihood estimation technique is used to estimate the model parameters. A simulation is carried out to examine the accuracy of the maximum likelihood estimates. The importance of the HLLLW is exemplified by two real life datasets.

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References

[1] Afify, A. Z., Kumar, D., and Elbatal, I. (2020). Marshall–Olkin power generalized Weibull distribution with applications in engineering and medicine. Journal of Statistical Theory and Applications, 19(2): 223-237.

[2] Ahmed, M. T., Khaleel, M. A., and Khalaf, E. K. (2020). The new distribution (Topp Leone Marshall Olkin-Weibull) properties with an application. Periodicals of Engineering and Natural Sciences, 8(2): 684-692.

[3] Anwar, A., and Bibi, A. (2018). The half-logistic generalized Weibull distribution. Journal of Probability and Statistics, Volume 2018, Article ID 8767826, 12 pages, https://doi.org/10.1155/2018/8767826

[4] Anwar, M., and Zahoor, J. (2018). The half-logistic Lomax distribution for lifetime modeling. Journal of Probability and Statistics, Volume 2018, Article ID 3152807, 12 pages, https://doi.org/10.1155/2018/3152807.

[5] Aryal, G. R., Ortega, E. M., Hamedani, G. G., and Yousof, H. M. (2017). The Topp-Leone generated Weibull distribution: Regression model, characterizations and applications. International Journal of Statistics and Probability, 6(1): 126-141.

[6] Asgharzadeh, A., Nadarajah, S., and Sharafi, F. (2016). Weibull Lindley distribution. Revstat-Statistical Journal, 16(1): 87-113.

[7] Barlow, R. E., Towland, R. H., and Freeman, T. (1984). A Bayesian analysis of stress-rupture life of kevlar 49/epoxy spherical pressure vessels. Proceedings of the Canadian Conference on Applied Statistics.

[8] Bebbington M., Lai C. D., and Zitikis R. (2007). A flexible Weibull ex-tension. Reliability Engineering System Safety, 92: 719-726.

[9] Chambers, J., Cleveland, W., Kleiner, B., and Tukey, J. (1983). Graphical methods for data analysis. Chapman and Hall, London.

[10] Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. Journal of Quality Technology, 27: 154-161.

[11] Cordeiro, G, M., Afify, A, Z., Yousof, H, M., Cakmakyapan, S., and Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. Annals of the Brazilian Academy of Sciences, 90(3): 2579-2598.

[12] Cordeiro, G. M., Alizadeh, M., and Marinho, P. R. D. (2016). The TypeI half-logistic family of distributions. Journal of Statistical Computation and Simulation, 86(4): 707–728.

[13] Cordeiro, G. M., Ortega E. M., and Nadarajah S. (2010). The Kumaraswamy Weibull distribution with applications to failure data. Journal of Franklin Institute, 347: 1399-1429.

[14] Dimitrakopoulou, T., Adamidis, K., and Loukas, S. (2007). A lifetime distribution with an upside-down bathtub-shaped hazard function. IEEE Transactions on Reliability, 56(2): 308-311.

[15] Elbatal, I. (2011). Exponentiated modified Weibull distribution. Economic Quality Control. 26: 189-200.

[16] Famoye, F., Lee, C., and Olumolade, O. (2005). The Beta Weibull Distribution. Journal of Statistical Theory and Applications, 4(2): 121-136.

[17] Ibrahim, M., and Yousof, H. (2020). Transmuted Topp-Leone Weibull lifetime distribution: Statistical properties and different method of estimation. Pakistan Journal of Statistics and Operation Research, 501-515.

[18] Lai, C. D., Xie, M., and Murthy, D. N. P. (2003). A modified Weibull distribution. IEEE Transactions on Reliability, 52(1): 33-37.

[19] Lai, C. D., Xie, M., and Murthy, D. N. P. (2005). Reply to on some recent modifications of Weibull distribution. IEEE Transactions on Reliability, 54(4): 563.

[20] Lemonte, A. J. (2014). The Beta log-logistic distribution. Brazilian Journal of Probability and Statistics, 28: 313-332. https://doi.org/10.1214/12-BJPS209.

[21] Mudholkar, G. S., and Srivastava D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure data. IEEE Transactions on Reliability, 43: 299-302.

[22] Muhammad, M., and Yahaya, M. A. (2017). The half logistic-Poisson distribution. Asian Journal of Mathematics and Applications, 1-15.

[23] Murthy, D. N. P., Xie, M., and Jiang, R. (2004). Weibull models. John Wiley, New York, doi:10.1002/047147326X.

[24] Nofal, Z. M., Afify, A. Z., Yousof, H. M., Granzotto, D. C., and Louzada, F. (2016). Kumaraswamy transmuted exponentiated additive Weibull distribution. International Journal of Statistics and Probability, 5(2): 78-99.

[25] Okasha, H. M., Basheer, A. M., and El-Baz, A. H. (2020). Marshall–Olkin Extended Inverse Weibull Distribution: Different Methods of Estimations. Annals of Data Science, 1-16.

[26] Oluyede, B., Foya, S., Warahena-Liyanage, G., and Huang, S. (2016). The log-logistic weibull distribution with applications to lifetime data. Austrian Journal of Statistics, 45: 43–69.

[27] Pal, M., Ali, M. M.,Woo, J. (2006). Exponentiated Weibull distribution. STATISTICA, anno LXVI, 2.

[28] Rényi, A. (1960). On measures of Entropy and information. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1: 547 - 561.

[29] Silva, G. O., Ortega, E. M. M., Cordeiro, G. M. (2010). The Beta modified Weibull distribution, lifetime data analysis, 16: 409-430.

[30] Xie, M., Tang, y., and T. N. Goh. (2002). A modified Weibull extension with bathtub-shaped failure rate function. Reliability Engineering and System Safety, 76(3): 279–285.

[31] Zakerzadeh, H., and Dolati, A. (2009). Generalized Lindley distribution. Journal of Mathematical Extension, 3(2): 13-25.

A. Terminologies

- Distribution
- Maximum likelihood estimation
- Simulation
- Rényi entropy

B. R Code: Pdf plot of HLLLoGW

```
f1=function(x,alpha,c,beta){
 y=(2*exp(-alpha*x^beta)*((1+x^c)^(-1))*((((1+x^c)^(-1))*c*(x^(c-1)))+
      alpha*beta*(x^(beta-1)))*(1+(1+x^c)*exp(-alpha*x^beta))^(-2)  )
      /(1-((1-((1+x^c)^(-1))**exp(-alpha*x^beta))/(1+((1+x^c)^(-1))*exp
      (-alpha*x^beta))))
return(y)
}
x=seq(0,1.5,by=0.001)
y1=f1(x,2.0,5.0,3.2)
plot(x,y1,ylim=c(0,3),col=2,'l',lwd=2,xlab="x",ylab="density")
y2=f1(x,1.2,0.9,0.6)
lines(x,y2,col=3,lwd=2)
y3=f1(x,0.1,5.2,1.0)
lines(x,y3,col=4,lwd=2)
y4=f1(x,4.0,1.0,1.2)
lines(x,y4,col=5,lwd=2)
legend("topright",c(
   expression(paste(alpha,'=2.0,',c,'=5.0,',beta,'=3.2')),
   expression(paste(alpha,'=1.2,',c,'=0.9,',beta,'=0.6')),
   expression(paste(alpha,'=0.1,',c,'=5.2,',beta,'=1.0')),
   expression(paste(alpha,'=4.0,',c,'=1.0,',beta,'=1.2'))),col=c
       (2,3,4,5),lwd=c(2,2,2,2))
```

C. R Code: Application of HLLLoGW

```
######
rm(list=ls())
library(stats4)
library(bbmle)
library(stats)
library(numDeriv)
library('bbmle')
x<-c(0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79,
     1.80,1.94,2.38,2.40,2.87,2.99,3.14,3.17,4.72,5.09)
hist(x)
```

```
HLLLoGW_LL<- function(alpha, beta, c) {
  -sum(log(2*exp(-alpha*x^beta)*((1+x^c)^(-1))*(((1+x^c)^(-1))*c*(x^(c
      -1))+alpha*beta*(x^(beta-1)))*(1+((1+x^c)^(-1))*exp(-alpha*x^beta
      ))^(-2)))
}
main.result<-mle2(HLLLoGW_LL, hessian = NULL, start=list(alpha=0.082996,
    beta=0.008799878 ,c=19.899827),optimizer="nlminb",lower=0)
summary(main.result)
```

D. R Code: Goodness-of-Fit for HLLLoGW

```
rm(list=ls())
install.packages("stats4")
install.packages("AdequacyModel")
install.packages("bbmle")
install.packages("stats")
library(stats4)
library(AdequacyModel)
library(bbmle)
library(stats)

air<-c(0.03,0.12,0.22,0.35,0.73,0.79,1.25,1.41,1.52,1.79,
    1.80,1.94,2.38,2.40,2.87,2.99,3.14,3.17,4.72,5.09)


# define HLLLoGW pdf
HLLLoGW_pdf<-function(parameter,x){
  alpha=parameter[1]
 beta=parameter[2]
c=parameter[3]
2*exp(-alpha*x^beta)*((1+x^c)^(-1))*(((1+x^c)^(-1))*c*(x^(c-1))+alpha*
    beta*(x^(beta-1)))*(1+((1+x^c)^(-1))*exp(-alpha*x^beta))^(-2)
}
# define HLLLoGW cdf
HLLLoGW_cdf<-function(parameter,x){
  alpha=parameter[1]
     beta=parameter[2]
c=parameter[3]
(1-((1+x^c)^(-1))*exp(-alpha*x^beta))/(1+((1+x^c)^(-1))*exp(-alpha*x^
    beta))
}
goodness.fit(pdf=HLLLoGW_pdf, cdf= HLLLoGW_cdf, mle=c(0.061398 ,
    2.405721 ,          0.822303  ),data=air ,method = "BFGS",domain = c
    (0,100),lim_inf = c(0,0,0,0),
                lim_sup = c(100,100,100,100))
```

E. R Code: Kurtosis plot

```
            ##Kurtosis vs c
plot.new()
plot.window(xlim = c(0.1,0.4),ylim = c(0,0.02),main="plot of kurtosis"
            ,xlab="c",ylab="ck")
axis(1)
axis(2)
box()
x=c(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8)
y=c( 0.01211349, 0.01364684, 0.01518482, 0.01672691, 0.01827261,
   0.01982138, 0.02137271, 0.02292608)
lines(x,y, col='2',lwd='2')
y=c(  0.002429468 ,0.003602263 ,0.004748522, 0.005868665 ,0.006963140,
   0.008032420 ,0.009077002 ,0.01009740)
lines(x,y, col='3',lwd='2')
y=c( 0.001634351 ,0.002457769, 0.003285858 ,0.004118504 ,0.004955588,
   0.005796988, 0.006642579, 0.007492230)
lines(x,y, col='4',lwd='2')
y=c(   0.01075547, 0.01275565, 0.01475171, 0.01674312, 0.01872934,
   0.02070988 ,0.02268424 ,0.02465194)
lines(x,y, col='5',lwd='2')
legend("topleft",pch = 1,cex = 0.8,c(
  expression(paste(alpha,'=0.7,',beta,'=1.0')),
  expression(paste(alpha,'=0.9,',beta,'=2.0')),
  expression(paste(alpha,'=0.5,',beta,'=1.0')),
  expression(paste(alpha,'=1.0,',beta,'=0.8'))),
  col=c(2,3,4,5),lwd = c(2,2,2,2))
title(main="Plot of kurtosis vs c",xlab="c",ylab="CK")
```

F. R Code: Skewness plot

```
        ##Skewness vs c
plot.new()
plot.window(xlim = c(0.1,0.4),ylim = c(0,0.02),main="plot of skewness"
            ,xlab="c",ylab="cs")
axis(1)
axis(2)
box()
x=c(0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8)
y=c(0.01338350, 0.01510334, 0.01682886, 0.01855935, 0.02029410,
   0.02203238, 0.02377347, 0.02551666)
lines(x,y, col='2',lwd='2')
y=c(  0.002678034, 0.003974739, 0.005238790, 0.006470775 ,0.007671325,
   0.008841107, 0.009980818 ,0.01109117)
lines(x,y, col='3',lwd='2')
```

```
y=c( 0.001806370, 0.002723436 ,0.003646267, 0.004574705 ,0.005508585,
    0.006447739, 0.007391992, 0.008341165)
lines(x,y, col='4',lwd='2')
y=c( 0.01185471, 0.01408377, 0.01630757, 0.01852538, 0.02073650,
    0.02294024 ,0.02513592, 0.02732291)
lines(x,y, col='5',lwd='2')
legend("topleft",pch = 1,cex = 0.8,c(
  expression(paste(alpha,'=0.7,',beta,'=1.5')),
  expression(paste(alpha,'=0.2,',beta,'=1.5')),
  expression(paste(alpha,'=0.2,',beta,'=0.1')),
  expression(paste(alpha,'=0.5,',beta,'=0.6'))),
  col=c(2,3,4,5),lwd = c(2,2,2,2))
title(main="Plot of skewness vs c",xlab="c",ylab="CS")
```

G. Elements of Score Vector

Elements of the score vector are given by

$$\frac{\partial \ell_n}{\partial \alpha} = -\sum_{i=1}^{n} x_i^{\beta} + \sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{\left[(1+x_i^c)^{-1} c x_i^{c-1} + \alpha\beta x_i^{\beta-1}\right]} + 2\sum_{i=1}^{n} \frac{(1+x_i^c)^{-1} x_i^{\beta} e^{-\alpha x_i^{\beta}}}{\left[1+(1+x_i^c)^{-1} e^{-\alpha x_i^{\beta}}\right]}, \tag{G.1}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial c} &= -\sum_{i=1}^{n} \frac{x_i^c \ln(x_i)}{(1+x_i^c)} + \sum_{i=1}^{n} \frac{\left[x_i^{c-1} + c x_i^{c-1}\ln(x_i)\right](1+x_i^c) - x_i^c \ln(x_i) c x_i^{c-1}}{(1+x_i^c)^2\left[(1+x_i^c)^{-1} c x_i^{c-1} + \alpha\beta x_i^{\beta-1}\right]} \\ &+ 2\sum_{i=1}^{n} \frac{(1+x_i^c)^{-2} x_i^c \ln(x_i) e^{-\alpha x_i^{\beta}}}{\left[1+(1+x_i^c)^{-1} e^{-\alpha x_i^{\beta}}\right]},\end{aligned} \tag{G.2}$$

and

$$\begin{aligned}\frac{\partial \ell_n}{\partial \beta} &= \sum_{i=1}^{n} \frac{\alpha\left[x_i^{\beta-1} + \beta x_i^{\beta-1}\ln(x_i)\right]}{\left[(1+x_i^c)^{-1} c x_i^{c-1} + \alpha\beta x_i^{\beta-1}\right]} + 2\sum_{i=1}^{n} \frac{(1+x_i^c)^{-1}\alpha x_i^{\beta}\ln(x_i) e^{-\alpha x_i^{\beta}}}{\left[1+(1+x_i^c)^{-1} e^{-\alpha x_i^{\beta}}\right]} \\ &- \alpha\sum_{i=1}^{n} x_i^{\beta}\ln(x_i).\end{aligned} \tag{G.3}$$