Generalised bi-ideals in ordered ternary semigroups

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Abstract. The aim of this paper is to introduce a new concept of an ordered \((m, (p, q), n)\)-bi-ideal of an ordered ternary semigroup. Some classical results in ordered ternary semigroups are given. We also consider the minimal ordered \((m, (p, q), n)\)-bi-ideals in ordered ternary semigroups. In particular, the \((m, (p, q), n)\)-bi simple ordered ternary semigroups are defined and some of their properties are explored. As a consequence, we will show that the regular ordered ternary semigroups can be characterized by using various generalised ideals.

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1. Introduction and preliminaries

A ternary semigroup is a particular case of an \(n\)-ary semigroup for \(n=3[2]\). In the literature, there are several applications of ternary structures in physics described by R. Kerner [13]. Obviously, ternary semigroups are universal algebra with one associative operation.

The notion of bi-ideal in semigroup was first initiated by R.A. Good and D.R. Hughes [12]. As a generalization of bi-ideals in semigroups, the concept of the \((m, n)\)-ideal in semigroups was given by S. Lajos. In addition, M.Y. Abbasi and Abul Basar also considered the generalized bi-\(\Gamma\)-ideals, prime, semiprime and irreducible generalized bi-\(\Gamma\)-ideals in \(\Gamma\)-semigroups [9]. The ideal theory in ternary semigroup and ternary near rings was established by F.M. Sioson[4] and he also introduced the notion of regular ternary semigroups and characterized these semigroups by using the notion of quasi-ideals. V.N. Dixit and S. Dewan [21] introduced and studied the properties of (left, right, lateral, quasi)-bi-ideals in ternary semigroups. Shum et. al. [16] defined the concept of singular ternary semirings and studied some radical classes related to singular ideals. M.K. Dubey and S. Anuradha [7] introduced generalized quasi-ideals and generalized bi-ideals in a ternary semigroup.


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ordered bi-ideals in ordered ternary semigroups, as a generalization of the notion of ordered left, right, lateral ideals defined by lampan in an ordered ternary semigroup [1]. They also defined regular ordered ternary semigroup. M.Y. Abbasi et. al. [10] further studied generalised quasi-ideals in ordered ternary semigroups.

We first give the definition of a ternary semigroup and cite some examples with the following definitions.

Definition 1.1. ([21]) A non-empty set $S$ is called a ternary semigroup if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \to [x_1x_2x_3]$, satisfying the following identity:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]],$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

In this paper, we denote $[x_1x_2x_3]$ by $x_1x_2x_3$.

Example 1.2. ([21]) Let $S = \{-i, 0, i\}$. Then $S$ is a ternary semigroup under the multiplication over complex number while $S$ is not a (binary) semigroup under complex number multiplication.

Example 1.3. Let $S = \mathbb{Z}^-$. Then $S$ is a ternary semigroup while $S$ is not a semigroup under the multiplication.

For non-empty subsets $A$, $B$ and $C$ of a ternary semigroup $S$, let $ABC := \{abc : a \in A, b \in B$ and $c \in C\}$. If $A = \{a\}$, then we also write $\{a\}BC$ as $aBC$, and similarly if $B = \{b\}$ or $C = \{c\}$ or $A = \{a\}$ and $B = \{b\}$ or $A = \{a\}$ and $C = \{c\}$ or $B = \{b\}$ and $C = \{c\}$.

We now consider the ordered ternary semigroups.

Definition 1.4. ([1]) A partially ordered ternary semigroup $S$ is called an ordered ternary semigroup if for any $x_1, x_2, x_3, x_4 \in S$, $x_1 \leq x_2$ implies $[x_1x_3x_4] \leq [x_2x_3x_4], [x_3x_1x_4] \leq [x_3x_2x_4]$ and $[x_3x_1x_2] \leq [x_3x_4x_2]$.

Example 1.5. Let $S = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{N}_0 \right\}$, where $\mathbb{N}_0$, the set of non-negative integers is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}_0}$ is "less than or equal to". Now we define partial order relation $\leq_S$ on $S$ by, for any $A, B \in S$,

$$A \leq_S B \text{ if and only if } a_{ij} \leq_{\mathbb{N}_0} b_{ij}, \text{ for all } i \text{ and } j.$$  

It is easy to see that $S$ is an ordered ternary semigroup under usual multiplication of matrices over $\mathbb{N}_0$ with partial order relation $\leq_S$.

Let $S$ be an ordered ternary semigroup. For $H \subseteq S$ we denote $(H)$ the subset of $S$ defined by

$$(H) = \{s \in S \mid s \leq h, \text{ for some } h \in H\}$$

Theorem 1.6. ([20]) Let $S$ be an ordered ternary semigroup, then the following hold

1. $A \subseteq (A)$, for all $A \subseteq S$.
2. If $A \subseteq B \subseteq S$, then $(A) \subseteq (B)$.
3. $(A] = (A)$, for all $A \subseteq S$.
4. $(A\{B\}C) \subseteq (ABC)$, for all $A, B, C \subseteq S$. 
Definition 1.7. A non-empty subset $T$ of an ordered ternary semigroup $S$ is called an ordered ternary subsemigroup of $S$, if $TTT \subseteq T$ and $|T| = T$.

Definition 1.8. ([20]) An ordered ternary semigroup $S$ is said to be regular if for each $a \in S$ there exists $x \in S$ such that $a \leq axa$, that is, an ordered ternary semigroup $S$ is regular if for each, $a \in S$, $a \in \{aSa\}$.

We now study ordered ideals of an ordered ternary semigroup.

Definition 1.9. A non-empty subset $I$ of $S$ is called an ordered left (respectively, an ordered right, an ordered lateral) ideal of $S$ if

1. $SSI \subseteq I$ (respectively $ISS \subseteq I$, $SIS \subseteq I$) and

2. For $a \in I$, $b \in S$ such that $b \leq a$ implies $b \in I$, that is $(I] = I$.

Definition 1.10. A non-empty subset $I$ of $S$ is called a two-sided ordered ideal of $S$ if $I$ is both an ordered left and an ordered right ideal of $S$.

Example 1.11. In Example 1.5. Let $I = \left\{ \begin{pmatrix} 0 & 0 & p \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix} : p, q \in \mathbb{N}_0 \right\}$. Then $I$ is a two-sided ordered ideal of $S$.

Definition 1.12. A non-empty subset $I$ of $S$ is called an ordered ideal if $I$ is an ordered left, an ordered right and an ordered lateral ideal of $S$.

Example 1.13. In Example 1.5. Let $J = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} : m \in \mathbb{N}_0 \right\}$. Then $J$ is an ordered ideal of $S$.

We denote by $L(A)$, $R(A)$, $M(A)$ and $I(A)$ the ordered left ideal, ordered right ideal, ordered lateral ideal and ordered ideal of an ordered ternary semigroup $S$, respectively generated by a non-empty subset $A$ of $S$, that is the least, with respect to the inclusion relation, ordered left ideal, ordered right ideal, ordered lateral ideal and ordered ideal of $S$, respectively containing $A$. As usual $L(A)$ (respectively $R(A)$, $M(A)$ and $I(A)$) coincides with the intersection of all ordered left ideals (respectively ordered right ideals, ordered lateral ideals and ordered ideals) of $S$ containing $A$, $I(A)$ is the intersection of all ordered ideals of $S$ containing $A$. Following properties are observed.

$L(A) = (A \cup SSA]$,
$R(A) = (A \cup ASS]$,
$M(A) = (A \cup SAS \cup SSASS]$, 
$I(A) = (A \cup SSA \cup ASS \cup SAS \cup SSASS]$, $A \subseteq S$.

For $\{a\}$, we write $L(a)$, $R(a)$, $M(a)$ and $I(a)$ simply for $L(\{a\})$, $R(\{a\})$, $M(\{a\})$ and $I(\{a\})$ respectively and we call them the principal ordered left ideal, the principal ordered right ideal, principal ordered lateral ideal and principal ordered ideal of $S$ respectively generated by $a \in S$.

For any $a \in S$, $(SSa]$, $(SaS]S$ and $(aSS]$ are ordered left, ordered lateral and ordered right ideal of $S$ respectively. Moreover for any $a \in S$ the set $(SSaS]$ is an ordered ideal of $S$. 
Definition 1.14. ([20]) A non-empty subset $Q$ of $S$ is called an ordered quasi ideal of $S$ if

(i) $(QSS) \cap (SQS \cup SSQSS) \cap (SSQ) \subseteq Q$ and

(ii) For $a \in Q$ and $b \in S$ such that $b \leq a$ implies $b \in Q$. i.e. $(Q) = Q$.

Definition 1.15. ([20]) A non-empty subset $B$ of $S$ is called an ordered bi-ideal of $S$ if

(i) $BSBSB \subseteq B$ and

(ii) For $a \in B$, $b \in S$ such that $b \leq a$ implies $b \in B$. i.e. $(B) = B$.

2. Generalised bi-ideal or an ordered $(m, (p, q), n)$-bi-ideal

In this section, we introduce the concept of a generalised bi-ideal or an ordered $(m, (p, q), n)$-bi-ideal in an ordered ternary semigroup and give out some classical results.

Definition 2.1. A ternary subsemigroup $B$ of an ordered ternary semigroup $S$ is called a generalised bi-ideal or an ordered $(m, (p, q), n)$-bi-ideal of $S$ if

(1) $B(SS)^{m-1}S^pBS^q(SS)^{n-1}B \subseteq B$, where $m, n, p, q$ are positive integers and $p + q$ is even,

(2) $(B) = B$.

Example 2.2. In the Examples 1.11 and 1.13, $I$ and $J$ are ordered $(m, (p, q), n)$-bi-ideals of $S$.

Remark 2.3. Every ordered bi-ideal of an ordered ternary semigroup $S$ is an ordered $(1, (1, 1), 1)$-bi-ideal of $S$. But an ordered $(m, (p, q), n)$-bi-ideal of an ordered ternary semigroup $S$ need not be an ordered bi-ideal of $S$.

Example 2.4. Let $S = \left\{ \begin{pmatrix} 0 & 0 & 0 & a \\ b & 0 & c & d \\ e & f & 0 & g \\ h & 0 & 0 & 0 \end{pmatrix} : a, b, c, d, e, f, g, h \in \mathbb{N}_0 \right\}$, where $\mathbb{N}_0$, the set of non-negative integers is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq S$ is "less than or equal to". Now we define partial order relation $\leq S$ on $S$ by

For any $A, B \in S$, $A \leq S B$ if and only if $a_{ij} \leq S b_{ij}$, for all $i$ and $j$.

Then it is easy to verify that $S$ is an ordered ternary semigroup under usual multiplication of matrices over $\mathbb{N}_0$ with partial order relation $\leq S$.

Let $B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} : b, c, d \in \mathbb{N}_0 \right\}$. Then $B$ is an ordered $(1, (2, 2), 1)$-bi-ideal of $S$ but $B$ is not an ordered $(1, (1, 1), 1)$-bi-ideal of $S$ because $BSBSB = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & d & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix} : b, c, d, e \in \mathbb{N}_0 \right\} \not\subseteq B$. 

Theorem 2.5. Let $S$ be an ordered ternary semigroup and $\{B_i : i \in I\}$, (where $I$ is an index set) be the family of ordered $((m, (p, q), n)\text{-bi-ideals}$ of $S$ such that $\bigcap_{i \in I} B_i \neq \emptyset$. Then $\bigcap_{i \in I} B_i$ is an ordered $(m, (p, q), n)\text{-bi-ideal}$ of $S$.

Definition 2.6. [10] A ternary subsemigroup $Q$ of an ordered ternary semigroup $S$ is called a generalised quasi-ideal or an ordered $(m, (p, q), n)\text{-quasi-ideal}$ of $S$ if,

1. $(Q(SS)^m) \cap ((S^pQS^q \cup S^pSQ^qSS^q)) \cap ((SS)^nQ) \subseteq Q$, where $m, n, p, q$ are positive integers greater than zero and $p + q = \text{even}$,

2. $(Q) \subseteq Q$.

Theorem 2.7. Every ordered $(m, (p, q), n)\text{-quasi-ideal}$ of an ordered ternary semigroup $S$ is an ordered $(m, (p, q), n)\text{-bi-ideal}$ of $S$.

Proof. Suppose that $Q$ is an ordered $(m, (p, q), n)\text{-quasi-ideal}$ of $S$. Then

\[Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m - SS(QSS)^n) \subseteq (Q(SS)^m - SS(QSS)^n),\]

Similarly,

\[Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m - SS(QSS)^n) \subseteq (Q(SS)^m - SS(QSS)^n),\]

Again,

\[Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m - SS(QSS)^n) \subseteq (Q(SS)^m - SS(QSS)^n),\]

So

\[Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m - SS(QSS)^n) \subseteq (Q(SS)^m - SS(QSS)^n),\]

Also,

\[Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m - SS(QSS)^n) \subseteq (Q(SS)^m - SS(QSS)^n),\]

Therefore $Q(SS)^m - SS(QSS)^n \subseteq (Q(SS)^m) \cap ((SS)^nQ) \subseteq Q$.

Also, since $Q$ is an ordered $(m, (p, q), n)\text{-quasi-ideal}$ of $S$. It implies $(Q) \subseteq Q$.

Hence $Q$ is an ordered $(m, (p, q), n)\text{-bi-ideal}$ of $S$.

Definition 2.8. [10] Let $S$ be an ordered ternary semigroup. Then a ternary subsemigroup

1. $R$ of $S$ is called an ordered $m$-right ideal of $S$ if $R(SS)^m \subseteq R$ and $(R) = R$.

2. $M$ of $S$ is called an ordered $(p, q)$-lateral ideal of $S$ if $(S^pMS^q \cup S^pSMSS^q) \subseteq M$ and $(M) = M$.

3. $L$ of $S$ is called an ordered $n$-left ideal of $S$ if $(SS)^nL \subseteq L$ and $(L) = L$.

where $m, n, p, q$ are positive integers and $p + q$ is an even positive integer.
Lemma 2.9. For any non-empty subset \( A \) of an ordered ternary semigroup \( S \)

1. \((A(SS)^m)^{-1}\) is an \( m \)-right ideal of \( S \)

2. \((SPASq)\) is an \((p,q)\)-lateral ideal of \( S \)

3. \(((SS)^n-1)A\) is an \( n \)-left ideal of \( S \).

Lemma 2.10. For any non-empty subset \( A \) of an ordered ternary semigroup \( S \),

\[
((A(SS)^m-1)[SPASq][(SS)^n-1,A])
\]

is an ordered \((m, p, q), n\)-bi-ideal of \( S \).

Lemma 2.11. If \( A \) is a ternary subsemigroup of an ordered ternary semigroup \( S \). Then

1. \((A(SS)^m-1 \cup A)\) is an ordered \( m \)-right ideal of \( S \)

2. \((SPASq \cup A)\) is an \((p,q)\)-lateral ideal of \( S \)

3. \(((SS)^n-1 \cup A)\) is an \( n \)-left ideal of \( S \).

Lemma 2.12. For any non-empty subset \( A \) of \( S \),

\[
(A(SS)^m-1SPASq(SS)^n-1A) \cup \left( \bigcup_{i=1}^{max(2m-1,p+q-1,2n-1)} A^i \right) \text{ (skip \( A^i \), if \( i \) is even)}
\]

is the smallest ordered \((m, p, q), n\)-bi-ideal of \( S \) containing \( A \).

Furthermore, for any \( a \in S \):

- \( R(a) = (a(SS)^m \cup \left( \bigcup_{i=1}^{2m-1} a^i \right) \) (skip \( a^i \), if \( i \) is even);
- \( M(a) = ((SPASq) \cup \left( \bigcup_{i=1}^{p+q-1} a^i \right) \) (skip \( a^i \), if \( i \) is even);
- \( L(a) = ((SS)^n \cup \left( \bigcup_{i=1}^{2n-1} a^i \right) \) (skip \( a^i \), if \( i \) is even).

\( B(a) = ((a(SS)^m)^{-1}[SPASq][(SS)^n^{-1}A] \cup \left( \bigcup_{i=1}^{max(2m-1,p+q-1,2n-1)} a^i \right) \) (skip \( a^i \), if \( i \) is even).

Proposition 2.13. If \( A \) is an ordered ternary subsemigroup of an ordered ternary semigroup \( S \) and \( B \) is an ordered \((m, p, q), n\)-bi-ideal of \( S \), then \( B \cap A \) is an ordered \((m, p, q), n\)-bi-ideal of \( A \).

Proof. Suppose that \( B \) is an ordered \((m, p, q), n\)-bi-ideal of \( S \) and \( A \) is an ordered ternary subsemigroup of \( S \). Then,

\[
(B \cap A)(AA)^m-1Ap(B \cap A)AA^{-1}(AA)^n-1(B \cap A) \subseteq B(SS)^m-1SP(BS^q(SS)^n-1B) \subseteq B
\]

and

\[
(B \cap A)(AA)^m-1Ap(B \cap A)AA^{-1}(AA)^n-1(B \cap A) \subseteq A(AA)^m-1ApAA^q(AA)^n-1A \subseteq A.
\]

Therefore, \((B \cap A)(AA)^m-1Ap(B \cap A)AA^{-1}(AA)^n-1(B \cap A) \subseteq B \cap A.

Now, \( B \cap A \subseteq A \Rightarrow (B \cap A) \subseteq [A] = A \) and \( B \cap A \subseteq B \Rightarrow (B \cap A) \subseteq [B] = B \). It implies \((B \cap A) \subseteq B \cap A \). Thus \( B \cap A \) is an ordered \((m, p, q), n\)-bi-ideal of \( A \).

Definition 2.14. Let \( A, B \) and \( C \) be three non-empty subsets of an ordered ternary semigroup \( S \),

then \((ABC)\) is said to be the ordered product of \( A, B \) and \( C \).
Proposition 2.15. If $B_1$, $B_2$ and $B_3$ are three ordered $(m, (p,q), n)$-bi-ideal of an ordered ternary semigroup $S$, then the ordered product $(B_1B_2B_3)$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

Proof. 

$$
((B_1B_2B_3)(SS)^{m-1}S^p(B_1B_2B_3)[SS]^n(B_1B_2B_3)) \subseteq ((B_1B_2B_3)(SS)^{m-1}S^p(B_1B_2B_3)[SS]^n(B_1B_2B_3))
$$

Now $((B_1B_2B_3)) = (B_1B_2B_3)$. Therefore $(B_1B_2B_3)$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

Proposition 2.16. If $Q_1$, $Q_2$ and $Q_3$ are three ordered $(m, (p,q), n)$-quasi-ideal of an ordered ternary semigroup $S$, then the ordered product $(Q_1Q_2Q_3)$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

Proposition 2.17. Let $R$, $M$, $L$ be an ordered $m$-right, an ordered $(p,q)$-lateral and an ordered $n$-left ideal of an ordered ternary semigroup $S$, respectively. Then the ordered product $(RML)$ of $S$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

Proof. Since every ordered $m$-right, ordered $(p,q)$-lateral and every ordered $n$-left ideal of $S$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$. Then by the proposition 2.15, $(RML)$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

3. Minimal $(m, (p,q), n)$-bi-ideal and $(m, (p,q), n)$-bi simple ordered ternary semigroup

In this section, we introduce the concept of minimal ordered $(m, (p,q), n)$-bi-ideal in ordered ternary semigroups and $(m, (p,q), n)$-bi simple ordered ternary semigroups. Here, we study some existing results of minimal ordered $(m, (p,q), n)$-bi-ideals and $(m, (p,q), n)$ simple ordered ternary semigroups.

Definition 3.1. An ordered $(m, (p,q), n)$-bi-ideal $B$ of an ordered ternary semigroup $S$ is said to be minimal if it does not properly contain any ordered $(m, (p,q), n)$-bi-ideal of $S$.

For an ordered $(m, (p,q), n)$-bi-ideal of an ordered ternary semigroup, we have the following theorem.

Theorem 3.2. Let $S$ be an ordered ternary semigroup and $B$ be an ordered $(m, (p,q), n)$-bi-ideal of $S$. Then $B$ is minimal if and only if $B$ is the ordered product of some minimal ordered $m$-right ideal $R$, minimal ordered $(p,q)$-lateral ideal $M$, minimal ordered $n$-left ideal $L$ of $S$.

Proof. Suppose $B$ is a minimal ordered $(m, (p,q), n)$-bi-ideal of $S$ and let $b \in B$. Then by the Lemmas 2.9 and 2.10, $(b(SS)^{m-1})$ is an ordered $m$-right ideal, $(SSq^p)$ is an ordered $(p,q)$-lateral ideal, $(SS)^{n-1}b$ is an ordered $n$-left ideal and $(b(SS)^{m-1})(SSq^p)(SS)^{n-1}b]$ is an ordered $(m, (p,q), n)$-bi-ideal of $S$.

Now 

$$
((b(SS)^{m-1})(SSq^p)(SS)^{n-1}b)] \subseteq ((b(SS)^{m-1})(SSq^p)(SS)^{n-1}b], \text{ by theorem 1.6}
$$

$= (b(SS)^{m-1})(SSq^p)(SS)^{n-1}b], \text{ by theorem 1.6}
$$

$\subseteq (B(SS)^{m-1})(SSq^p)(SS)^{n-1}B]
$$

$\subseteq (B(SS)^{m-1})(SSq^p)(SS)^{n-1}B]
$$

$\subseteq B.
$$
As \( B \) is minimal, we have \((b(SS)^{-1}](S^{p}bS^{q}][(SS)^{-1}]b]) = B. 
Now we have to show that \((SS)^{-1}b\) is a minimal ordered \( n \)-left ideal of \( S \). Let \( L \) be an ordered \( n \)-left ideal of \( S \) contained in \((SS)^{-1}b\).

Then
\[
((b(SS)^{-1}](S^{p}bS^{q}][SS)^{-1}])b] = B.
\]

As \((b(SS)^{-1}](S^{p}bS^{q}][SS)^{-1}])b] \) is an ordered \( (m, p, q, n) \)-bi-ideal of \( S \) and \( B \) is minimal, therefore \((b(SS)^{-1}](S^{p}bS^{q}][SS)^{-1}])b] = B. \) This implies \( B \subseteq L \)

Now
\[
((SS)^{-1}b] \subseteq (SS)^{-1}B \subseteq (SS)^{-1}L. 
\]

The above result implies \((SS)^{-1}b] = L. \) Therefore \((SS)^{-1}b] \) is a minimal ordered \( n \)-left ideal of \( S \).

Similarly, we can prove that \((b(SS)^{-1}]) \) is a minimal ordered \( m \)-right ideal of \( S \) and \( (S^{p}bS^{q}]\) is a minimal ordered \( (p, q) \)-lateral ideal of \( S \).

Conversely, suppose that \( B = (RML) \), for some minimal ordered \( m \)-right ideal \( R \), minimal ordered \( (p, q) \)-lateral ideal \( M \) and minimal ordered \( n \)-left ideal \( L \) of \( S \). So \( B \subseteq R, B \subseteq M \) and \( B \subseteq L \). Let \( B' \) be any ordered \( (m, p, q, n) \)-bi-ideal of \( S \) contained in \( B \). Then
\[
(B'(SS)^{-1}]) \subseteq (B(SS)^{-1}]) \subseteq (R(SS)^{-1}]) \subseteq R.
\]

Similarly \((S^{p}B'S^{q} \cup S^{p}SB'S^{q}] \subseteq M \) and \((SS)^{-1}B'] \subseteq L \)

Now
\[
((B'(SS)^{-1}](SS]) \subseteq ((B'(SS)^{-1}](SS]) by Th : 1.6 \subseteq (B'(SS)^{-1}](SS]) by Th : 1.6 \subseteq (B'(SS)^{-1}].
\]

Therefore \((B'(SS)^{-1}]) \) is an ordered \( m \)-right ideal of \( S \). Similarly \((S^{p}B'S^{q}]\) is an ordered \( (p, q) \)-lateral of \( S \) and \((SS)^{-1}B'] \) is an ordered \( n \)-left ideal of \( S \). Since \( R, M \) and \( L \) are minimal ordered \( m \)-right of \( S \), minimal ordered \( (p, q) \)-lateral of \( S \) and minimal ordered \( n \)-left ideal of \( S \), respectively. Therefore \((B'(SS)^{-1}]) = R, (S^{p}B'S^{q}] = M \) and \((SS)^{-1}B'] = L. \) Thus
\[
B = (RML) = ((B'(SS)^{-1}](S^{p}B'S^{q}]][SS)^{-1}B')] \subseteq ((B'(SS)^{-1}](S^{p}B'S^{q}][SS)^{-1}B']), by Th : 1.6 \subseteq (B'(SS)^{-1}](S^{p}B'S^{q}][SS)^{-1}B']), by Th : 1.6 \subseteq (B'(SS)^{-1}].
\]

i.e. \( B \subseteq B' \). Hence \( B = B' \). Consequently \( B \) is a minimal ordered \((m, p, q, n) \)-bi-ideal of \( S \).

**Theorem 3.3.** If \( S \) is an ordered ternary semigroup and it has a proper ordered \((m, p, q, n) \)-bi-ideal, then every proper ordered \((m, p, q, n) \)-bi-ideal of \( S \) is minimal if and only if the intersection of any two distinct proper ordered \((m, p, q, n) \)-bi-ideals is empty.
Proof. Assume that $B_1$ and $B_2$ are two distinct proper ordered $(m, (p, q), n)$-bi-ideals of $S$. Then $B_1$ and $B_2$ are minimal. If $B_1 \cap B_2 \neq \emptyset$, then $B_1 \cap B_2$ is an ordered $(m, (p, q), n)$-bi-ideal of $S$. Since $B_1$ and $B_2$ are minimal. Then $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$, it implies $B_1 = B_2$, which is a contradiction. Therefore $B_1 \cap B_2 = \emptyset$.

Converse is obvious.

**Theorem 3.4.** Let $S$ be an ordered ternary semigroup. Then an ordered $(m, (p, q), n)$-bi-ideal $B$ is minimal if and only if $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) = B$ for all $a \in B$.

**Proof.** Suppose that an ordered $(m, (p, q), n)$-bi ideal $B$ is minimal. Let $a \in B$. Then

$$((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) \subseteq ((B(SS)^{-1}[S^p B S^q][SS]^{n-1}B]) \subseteq B.$$  

By Lemma 2.10, we have $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a])$ is an ordered $(m, (p, q), n)$-bi ideal of $S$. As $B$ is minimal ordered $(m, (p, q), n)$-bi ideal of $S$. We have $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) = B$.

Conversely, suppose that $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) = B$ for all $a \in B$. Let $B'$ be any ordered $(m, (p, q), n)$-bi ideal of $S$ contained in $B$. Let $b \in B'$. By assumption, we have $((b(SS)^{-1}[S^p b S^q][SS]^{n-1}b]) = B$ for all $b \in B$.

Now

$$B = ((b(SS)^{-1}[S^p b S^q][SS]^{n-1}b]) \subseteq ((B'(SS)^{-1}[S^p B' S^q][SS]^{n-1}B']) \subseteq B'.$$

It implies $B \subseteq B'$. Thus $B = B'$. Hence $B$ is minimal ordered $(m, (p, q), n)$-bi ideal of $S$.

**Definition 3.5.** Let $S$ be an ordered ternary semigroup. Then $S$ is said to be $(m, (p, q), n)$-bi-simple if $S$ is the unique ordered $(m, (p, q), n)$-bi-ideal of $S$.

**Lemma 3.6.** Let $B$ be an ordered $(m, (p, q), n)$-bi-ideal of an ordered ternary semigroup $S$ and $J$ an ordered ternary subsemigroup of $S$. If $J$ is an $(m, (p, q), n)$-bi-simple ordered ternary semigroup such that $J \cap B \neq \emptyset$, then $J \subseteq B$.

**Proof.** Assume that $J$ is $(m, (p, q), n)$-bi-simple such that $J \cap B \neq \emptyset$, and let $a \in J \cap B$. By Lemma 2.10, $(a(JJ)^{-1}J^p a J^q(JJ)^{-1}a]$ is an ordered $(m, (p, q), n)$-bi-ideal of $J$. As $J$ is $(m, (p, q), n)$-bi-simple it implies that $(a(JJ)^{-1}J^p a J^q(JJ)^{-1}a]) = J$.

Therefore $J = (a(JJ)^{-1}J^p a J^q(JJ)^{-1}a]) \subseteq (B(SS)^{-1}S^p B S^q(SS)^{-1}B] \subseteq (B] \subseteq B$, So $J \subseteq B$.

**Lemma 3.7.** Let $S$ be an ordered ternary semigroup. Then the following statements are equivalent:

(i) $S$ is $(m, (p, q), n)$-bi-simple.

(ii) $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) = S$, for all $a \in S$.

(iii) $B(a) = S$ for all $a \in S$.

**Proof.** Since $S$ is $(m, (p, q), n)$-bi-simple and by Lemma 2.10, $(a(SS)^{-1}S^p a S^q(SS)^{-1}a]$ is an ordered $(m, (p, q), n)$-bi-ideal of $S$, Then $((a(SS)^{-1}[S^p a S^q][SS]^{n-1}a]) = S$, for all $a \in S$. Therefore (i) implies (ii).
Now
\[ B(a) = ((a(SS)^{m-1})[S^p aS^q][(SS)^{n-1}a] \cup \left( \bigcup_{i=1}^{\text{max}(2m-1,p+q-1,2n-1)} a^i \right)) \]
\[ = (\left( (a(SS)^{m-1})[S^p aS^q][(SS)^{n-1}a] \right) \cup \left( \bigcup_{i=1}^{\text{max}(2m-1,p+q-1,2n-1)} a^i \right)) \]
\[ = S \cup \left( \bigcup_{i=1}^{\text{max}(2m-1,p+q-1,2n-1)} a^i \right) \]
\[ = S, \text{ for all } a \in S. \]

Hence (ii) implies (iii).

If \( B \) is an ordered \((m, (p, q), n)\)-bi-ideal of \( S \) and \( a \in B \), then \( S = B(a) \subseteq B \subseteq S \). So \( S = B \).

Hence \( S \) is \((m, (p, q), n)\)-bi-simple. We have that (iii) implies (i).

**Theorem 3.8.** Let \( S \) be an ordered ternary semigroup and \( B \) be an ordered \((m, (p, q), n)\)-bi-ideal of \( S \). If \( B \) is a \((m, (p, q), n)\)-bi-simple ordered ternary semigroup, then \( B \) is a minimal ordered \((m, (p, q), n)\)-bi-ideal of \( S \).

**Proof.** Suppose \( B \) is an \((m, (p, q), n)\)-bi-simple ordered ternary semigroup. Then by the Lemma 3.7, \((a(BB)^{m-1})[B^p aB^q][(BB)^{n-1}a] = B, \forall a \in B.\) For all \( a \in B \), we have \( B = ((a(SS)^{m-1})[S^p aS^q][(SS)^{n-1}a]) \subseteq ((aSS)^{m-1})[S^p aS^q][(SS)^{n-1}a] \subseteq B \). Therefore \( B = ((aSS)^{m-1})[S^p aS^q][(SS)^{n-1}a] \) for all \( a \in B \). Hence by the Theorem 3.4, \( B \) is minimal.

4. Generalised bi-ideals in regular ordered ternary semigroups

In this section, we study some interesting properties of ordered \((m, (p, q), n)\)-bi-ideals in regular ordered ternary semigroups and characterize regular ordered ternary semigroups in terms of various generalised ideals.

**Theorem 4.1.** A ternary subsemigroup \( B \) of a regular ordered ternary semigroup \( S \) is an ordered \((m, (p, q), n)\)-bi-ideal of \( S \) if and only if \( B = (BSB) \).

**Proof.** Suppose that \( B \) is an ordered \((m, (p, q), n)\)-bi-ideal of a regular ordered ternary semigroup \( S \). Let \( b \in B \). Then there exists \( x \in S \) such that \( b \leq bxb \). This implies that \( b \in (BSB) \). Hence \( B \subseteq (BSB) \).

Now
\[ (BSB) \subseteq (BSB[S(BSB)]) \]
\[ = (BSB[S(BSB)]) \subseteq (BSB[S][BS(BSB)]) \]
\[ = (BSB[S][BS(BSB)]), \text{ as } B \text{ and } S \text{ are ordered} \]
\[ \subseteq (BSB[S][BSBSBSB]) \]
\[ \subseteq (BSB[S][BSBSBSB]), \text{ since } (A] = (A) \]
\[ \subseteq (BSBBSBSBSB) \]
\[ = (BSBBSBSBSB) \subseteq (B(SS)(BBS)(SS)B], \text{ by the property of regularity it is easy to show that } (BSB) \subseteq (B(SS)^{m-1}S^p BS^q(SS)^{n-1}B). \]

Given that \( B \) is an ordered \((m, (p, q), n)\)-bi-ideal of \( S \). Hence \( (BSB) \subseteq (B(SS)^{m-1}S^p BS^q(SS)^{n-1}B \subseteq (B[S(SS)^{m-1}S^p BS^q(SS)^{n-1}B \subseteq (B) \subseteq B. \) Therefore \( B = (BSB). \)

Conversely if \( B = (BSB) \), then \( B(SS)^{m-1}S^p BS^q(SS)^{n-1}B \subseteq B(SS)^{m-1}S^p BS^q(SS)^{n-1}B \subseteq BSB \subseteq (BSB) = B. \)

Now \( (B) = ((BSB]) = (BSB) = B. \) Hence \( B \) is an ordered \((m, (p, q), n)\)-bi-ideal of \( S \).
Proposition 4.2. If $S$ is an regular ordered ternary semigroup and $B$ is an ordered $(m, (p, q), n)$-bi-ideal of $S$, then

$$(BS)^{m-1}SpBS^q(SS)^{n-1}B = B$$

Proof. Proof is similar to the Theorem 4.1.

Corollary 4.3. Let $S$ be a regular ordered ternary semigroup. If $Q$ is an ordered $(m, (p, q), n)$-quasi ideal of $S$, then

$$(Q(SS)^{m-1}SpQS^q(SS)^{n-1}Q| = Q$$

Proof. Proof is similar to the Theorem 4.1.

Theorem 4.4. Let $S$ be an ordered ternary semigroup. Then $S$ is regular if and only if for every ordered $(m, (p, q), n)$-bi-ideal $B$, an ordered $(p, q)$-lateral ideal $M$ and an ordered $n$-left ideal $L$ of $S$, we have $B \cap M \cap L \subseteq (BML]$.

Proof. If $S$ is regular, then for any $a \in B \cap M \cap L$, $\exists x \in S$ s.t.

$$a \leq axaxaxaxaxaxa$$

$$a \leq axaxaxaxaxaxaxaxaxa$$

$$a \leq (axaxaxaxaxaxa)(axa)$$

$$\in (BSSSBSSSB)(SMS)(SSL)$$

$$\in BML.$$

By the property of regularity it is easy to show that $a \in (B(SS)^{m-1}SpBS^q(SS)^{n-1}B)(SpMS^q)((SS)^{n-1}L)$. It implies $a \in (BML].$ Therefore $B \cap M \cap L \subseteq (BML].$

Conversely, suppose that $B \cap M \cap L \subseteq (BML]$ for any ordered $(m, (p, q), n)$-bi-ideal $B$, ordered $(p, q)$-lateral ideal $M$ and ordered $n$-left ideal $L$ of $S$. For any $a \in S$, $(a(SS)^{m-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i))$ is an ordered $(m, (p, q), n)$-bi-ideal of $S$ containing $a$, $(SpaS^q \cup (\bigcup_{i=1}^{p+q-1} a^i))$ is an ordered $(p, q)$-lateral ideal of $S$ containing $a$ and $(SS)^{n-1}a \cup (\bigcup_{i=1}^{2n-1} a^i)$ is an ordered $n$-left ideal of $S$ containing $a$. Then

$$a \in (a(SS)^{m-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i)) \cap$$

$$(a(SS)^{m-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i))(SpaS^q \cup (\bigcup_{i=1}^{p+q-1} a^i)) \subseteq \cap ((SS)^{n-1}a \cup (\bigcup_{i=1}^{2n-1} a^i))$$

$$((-a(SS)^{m-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i))((SS)^{n-1}a \cup (\bigcup_{i=1}^{2n-1} a^i)))$$

$$((-a(SS)^{m-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i))((SS)^{n-1}a \cup (\bigcup_{i=1}^{2n-1} a^i)))$$

$$((-aSS)^{n-1}SpaS^q(SS)^{n-1}a \cup (\bigcup_{i=1}^{\max(2m-1,p+q-1,2n-1)} a^i))((SS)^{n-1}a \cup (\bigcup_{i=1}^{2n-1} a^i)))$$

It implies $a \in (aSa].$ Hence $S$ is regular.

Theorem 4.5. For an ordered ternary semigroup $S$, the following statements are equivalent:

1. $S$ is regular;
Proof. Proof is similar to the above theorem and is hence omitted.

Theorem 4.6. Let $S$ be a regular ordered ternary semigroup. Then every ordered $(m, p, q, n)$-bi-ideal is an ordered $(m, p, q, n)$-quasi-ideal of $S$.

Proof. Let $B$ be an ordered $(m, p, q, n)$-bi-ideal of $S$. Let $a \in (B(SS)^m \cap (S^p BS^q \cup S^p SBSS^q)) \cap ((SS)^n B)$. Then $a \in (B(SS)^m \cap (S^p BS^q \cup S^p SBSS^q))$ and $a \in ((SS)^n B)$.

Now $a \in (B(SS)^m)$. Hence $a \leq b_1$, for some $b_1 \in B(SS)^m$, $a \leq b_2$ for some $b_2 \in S^p BS^q \cup S^p SBSS^q$ and $a \leq b_3$, for some $b_3 \in (SS)^n B$.

Since $S$ is a regular ordered ternary semigroup and $a \in S$, there exists $x \in S$ such that $a \leq axa \leq azaxa \leq b_1x_2x_3$, where

$$b_1x_2x_3 \in B(SS)^mS(S^p BS^q \cup S^p SBSS^q)S(SS)^n B = B(SS)^mS^p BS^q S(SS)^n B \cup B(SS)^mS^p SBSS^q S(SS)^n B \subseteq B(SS)^mS^p BS^q S(SS)^n B \cup B(SS)^mS^p SBSS^q S(SS)^n B \subseteq BSB \cup BSB = BSB.$$ 

Therefore $a \in (BSB) = B$, since $S$ is regular. It implies $a \in B$ and $(B) \subseteq B$. Hence $B$ is an ordered $(m, p, q, n)$-quasi-ideal of $S$.

Alternative method: Let $B$ be an ordered $(m, p, q, n)$-bi-ideal of $S$. Then $(B(SS)^{m-1})$, $(S^p BS^q \cup S^p SBSS^q)$ and $((SS)^{n-1} B)$ are an ordered $m$-right, an ordered $(p, q)$-lateral and an ordered $n$-left ideals of $S$ respectively. From Theorem 4.4, we find that if $S$ is a regular ordered ternary semigroup, then $R \cap M \cap L \subseteq (RML)$, for any ordered $m$-right ideal $R$, any ordered $(p, q)$-lateral ideal $M$ and any ordered $n$-left ideal $L$.

Now

$$((B(SS)^{m-1}) \cap (S^p BS^q \cup S^p SBSS^q)) \cap ((SS)^{n-1} B)$$

$$\subseteq ((B(SS)^{m-1})[S^p BS^q \cup S^p SBSS^q][((SS)^{n-1} B)])$$

$$\subseteq ((B(SS)^{m-1})[S^p BS^q \cup S^p SBSS^q])((SS)^{n-1} B)$$

$$= (B(SS)^{m-1})(S^p BS^q \cup S^p SBSS^q)(SS)^{n-1} B$$

$$\subseteq (B(SS)^{m-1}(S^p BS^q)(SS)^{n-1} B \cup B(SS)^{m-1}(S^p SBSS^q)(SS)^{n-1} B)$$

$$\subseteq (BSB \cup BSB) = BSB.$$ 

Consequently, $B$ is an ordered $(m, p, q, n)$-quasi-ideal of $S$.

References


