EURASIAN BULLETIN OF MATHEMATICS
EBM (2021), VOL. 4, NO. 2, 61-70

S-prime and S-weakly Prime Submodules

Emel Aslankarayigit Ugurlu

Department of Mathematics, Marmara University, Kadikoy, Istanbul, 34722, Turkey

Abstract. In this study, all rings are commutative with non-zero identity and all modules are considered to be unital. Let M be a left R-module. A proper submodule N of M is called an *S-weakly prime* submodule if $0_M \neq f(m) \in N$ implies that either $m \in N$ or $f(M) \subseteq N$, where $f \in S = End(M)$ and $m \in M$. Some results concerning S-prime and S-weakly prime submodules are obtained. Then we study S-prime and S-weakly prime submodules of multiplication modules. Also for R-modules M_1 and M_2, we examine S-prime and S-weakly prime submodules of $M = M_1 \times M_2$, where $S = S_1 \times S_2$, $S_1 = End(M_1)$ and $S_2 = End(M_2)$.

2010 Mathematics Subject Classifications: 06F10; 06F05

Key Words and Phrases: *S-prime* submodule, *S-weakly prime* submodule, multiplication module

1. Introduction

Throughout this paper R will denote a commutative ring with a non-zero identity and M is considered to be unital left R-module. A proper ideal P of R is said to be *prime* if $ab \in P$ implies $a \in P$ or $b \in P$, [3]. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [2]. A proper ideal P of R is said to be *weakly prime* if $0_R \neq ab \in P$ implies $a \in P$ or $b \in P$. Several authors have extended the notion of prime ideals to modules, see, for example, [5, 9, 11]. In [1], a proper submodule N of a module M over a commutative ring R is said to be *prime* submodule if whenever $rm \in N$ for some $r \in R, m \in M$, then $m \in N$ or $rM \subseteq N$. Then in [6], S. Ebrahimi and F. Farzalipour introduced weakly prime submodules over a commutative ring R as following: A proper submodule N of M is called *weakly prime* if for $r \in R$ and $m \in M$ with $0_M \neq rm \in N$, then $m \in N$ or $rM \subseteq N$. Clearly, every prime submodule of a module is a weakly prime submodule. However, since 0_M is always weakly prime, a weakly prime submodule need not be prime. Various properties of weakly prime submodules are considered in [6].

Now we define the concepts the residue of N by M. If N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is called the residue of N by M and it is denoted by $(N :_R M)$. In particular, $(0_M :_R M)$ is called the *annihilator* of M and denoted by $Ann(M)$, see [7]. If the annihilator of M equals to 0_R, then M is called a *faithful module.* Also, for a proper submodule N of M, the *radical* of N, denoted by $\sqrt{N}$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then $\sqrt{N} = M$, see [7].

An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R, see [7]. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$.

Email address: emelakyugurlu@gmail.com

So, if M is multiplication, $N = (N :_R M)M$, for every submodule N of M. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations of N and K. Note that, for $m, m' \in M$, by mm', we mean the product of Rm and Rm'. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$, see [1]. Also, if M is multiplication module, in Theorem 3.13 of [1], R. Ameri showed that $\sqrt{N} = \{m \in M : m^k \subseteq N$ for some positive integer $k\}$.

This paper is inspired by the notion of S-prime submodule which appears in [8, 4]. The authors defined the concept as following: A proper submodule N of an R-module M is said to be *S-prime* submodule of M if $f(m) \in N$ implies that either $m \in N$ or $f(M) \subseteq N$, where $f \in S = End(M)$ and $m \in M$. Every *S-prime* submodule is prime, see [8]. For more information one can examine [4].

In this study we introduce the concept of *S-weakly prime* submodule as following: A proper submodule N of an R-module M is said to be *S-weakly prime* submodule of M if $0_M \neq f(m) \in N$ implies that either $m \in N$ or $f(M) \subseteq N$, where $f \in S = End(M)$ and $m \in M$. Clearly, every *S-prime* submodule is an *S-weakly prime* submodule. In Proposition 2.1, it is obtained that every *S-weakly prime* submodule of an R-module M is a *weakly prime* submodule. However, we show that the opposite of Proposition 2.1 is not correct, see Example 2.1. Then we prove in Proposition 2.2 (Proposition 2.4) that N is an S-prime (S-weakly prime) submodule if and only if $f(K) \subseteq N$ ($0_M \neq f(K) \subseteq N$) implies that either $K \subseteq N$ or $f(M) \subseteq N$, where $f \in S = End(M)$ and K is a submodule of M. Also, we give characterizations of *S-prime* submodule and *S-weakly prime* submodule (Theorem 2.2, Theorem 2.1, recpectively). In Corollary 2.4, by the help of Proposition 2.6, it is proved that when N is an S-weakly prime submodule, then $(N :_R M)$ is an S-weakly prime ideal of R. In multiplication module, we obtain another characterizations for *S-prime* submodule and *S-weakly prime* submodule (Theorem 2.3, Theorem 2.4, recpectively). Among the other results, some properties of *S-prime* and *S-weakly prime* submodules in multiplication modules are obtained. Moreover, we characterize *S-prime* and *S-weakly prime* submodules of $M = M_1 \times M_2$ over R-module, where M_1, M_2 be R-modules, see Theorem 2.7, Theorem 2.8, Proposition 2.7, Proposition 2.8. Finally, we obtain a relation between *S-prime* and *S-weakly prime* submodules of $M = M_1 \times M_2$ over $R = R_1 \times R_2$-module, where M_1, M_2 are R_1-module and R_2-module, recpectively, see, Theorem 2.9 and Theorem 2.10.

2. S-prime and S-weakly Prime Submodules

Throughout this study $End(M)$ is denoted by S.

Definition 2.1. *A proper submodule N of an R-module M is said to be S-weakly prime submodule of M if $0_M \neq f(m) \in N$ implies that either $m \in N$ or $f(M) \subseteq N$, where $f \in S = End(M)$ and $m \in M$.*

It is clear that every *S-prime* submodule is an *S-weakly prime* submodule. However, since $\{0_M\}$ is an *S-weakly prime* submodule of M, then an *S-weakly prime* submodule may not be an *S-prime* submodule.

Note that if we consider any R as an R-module, then a proper ideal I of R is said to be *S-prime* (*S-weakly prime*) ideal if $f(a) \in I$ ($0_R \neq f(a) \in I$) implies that either $a \in I$ or $f(R) \subseteq I$, where $f \in S = End(R)$ and $a \in R$.

Proposition 2.1. *Every S-weakly prime submodule of an R-module M is a weakly prime submodule.*

Proof. Let N be an *S-weakly prime* submodule of an R-module M. Suppose that for some $r \in R$ and $m \in M$ such that $0_M \neq rm \in N$ and $m \notin N$. We show that $r \in (N :_R M)$.

Define $h : M \to M$ such that $h(x) = rx$ for all $x \in M$. Clearly, $h \in End(M)$. Since definition of h and our assumption, then $0_M \neq h(m) = rm \in N$. Then as $m \notin N$ and N is an *S-weakly prime* submodule, we get $h(M) \subseteq N$. Thus $h(M) = rM \subseteq N$, i.e., $r \in (N :_R M)$.

Note that generally a *weakly prime* submodule is not an *S-weakly prime* submodule. For this one can see the following example:

Example 2.1. *Let consider the submodule $N = 2\mathbb{Z} \oplus \mathbb{Z}$ of $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$. Since N is a maximal submodule, N is a prime, so weakly prime submodule. But N is not an S-weakly prime submodule. Indeed, let define $f : M \longrightarrow M$ such that $f((x, y)) = (y, x)$ for all $(x, y) \in M$. Then we get $f \in S = End(M)$. Thus we can easily see $0_M \neq f((1, 2)) = (2, 1) \in N$, but $(1, 2) \notin N$ and $f(M) = M \nsubseteq N$. Consequently, N is not S-weakly prime.*

Proposition 2.2. *Let M be an R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N is an S-prime submodule.

(ii) $f(K) \subseteq N$ implies that either $K \subseteq N$ or $f(M) \subseteq N$, where $f \in S$ and K is a proper submodule of M.

Proof. (1) $\Longrightarrow$ (2) : Let N be an S-prime submodule. Assume that $f(K) \subseteq N$ and $K \nsubseteq N$. Then there exists $k \in K - N$. Thus $f(k) \in f(K) \subseteq N$. Since N is S-prime, $f(M) \subseteq N$.

(2) $\Longrightarrow$ (1) : Let $f(m) \in N$. We show that either $m \in N$ or $f(M) \subseteq N$. Since $f(Rm) \subseteq N$, by our hypothesis we obtain either $Rm \subseteq N$ or $f(M) \subseteq N$. Consequently, $m \in N$ or $f(M) \subseteq N$.

Corollary 2.1. *Let M be an R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N is an S-prime submodule.

(ii) $f(Rm) \subseteq N$ implies that either $Rm \subseteq N$ or $f(M) \subseteq N$, where $f \in S$ and $m \in M$.

Proof. By Proposition 2.2.

Proposition 2.3. *Let M be an R-module and N be a proper submodule of M. Then N is an S-prime submodule if and only if $f^{-1}(N) = M$ or $f^{-1}(N) \subseteq N$, for all $f \in S$.*

Proof. $\Longrightarrow$: Let N be an S-prime submodule. Assume that $f(M) \subseteq N$. Then it is clear that $f^{-1}(N) = M$. So suppose that $f(M) \nsubseteq N$. Take $m \in f^{-1}(N)$. Then $f(m) \in N$. Since N is an S-prime submodule and $f(M) \nsubseteq N$, we have $m \in N$. Consequently, $f^{-1}(N) \subseteq N$.

$\Longleftarrow$: Assume that $f^{-1}(N) \subseteq N$ or $f^{-1}(N) = M$ for all $f \in End(M)$. Let $f(m) \in N$. If $f^{-1}(N) \subseteq N$, then $m \in f^{-1}(N) \subseteq N$, so we are done. On the other hand, if $f^{-1}(N) = M$, then we get $f(M) \subseteq N$. Thus N is an S-prime submodule.

Corollary 2.2. *The zero submodule $\{0_M\}$ of M is an S-prime submodule if and only if f is one-to-one, for all $0 \neq f \in S$.*

Proof. By Proposition 2.3.

Theorem 2.1. *Let M be an R-module and N be a proper submodule of M. For all $f \in S$, the followings are equivalent:*

(i) N *is an* S*-weakly prime submodule of* M.

(ii) $(N :_R f(x)) = (N :_R f(M)) \cup (0_M :_R f(x))$ *for all* $x \notin N$.

(iii) $(N :_R f(x)) = (N :_R f(M))$ *or* $(N :_R f(x)) = (0_M :_R f(x))$ *for all* $x \notin N$.

Proof. (1) $\Longrightarrow$ (2) : Assume that N is *S-weakly prime*. Let $r \in (N :_R f(x))$ and $x \notin N$. Then we get $rf(x) \in N$. If $rf(x) = 0_M$, then $r \in (0_M :_R f(x))$. Suppose that $rf(x) \neq 0_M$. Define $h : M \to M$ such that $h(m) = rf(m)$, for all $m \in M$. Clearly $h \in End(M)$, also $0_M \neq h(x) = rf(x) \in N$. Since N is an *S-weakly prime* submodule and $x \notin N$, then we obtain $h(M) \subseteq N$. Thus $h(M) = rf(M) \subseteq N$ and so $r \in (N :_R f(M))$.

(2) $\Longrightarrow$ (3) : Clear.

(3) $\Longrightarrow$ (1) : Suppose that $h \in End(M)$ and $m \notin N$ such that $0_M \neq h(m) \in N$. We prove that $h(M) \subseteq N$. Since $0_M \neq h(m)$, we get $(N :_R h(m)) \neq (0_M :_R h(m))$. Indeed, if $(N :_R h(m)) = (0_M :_R h(m))$, then we obtain $(N :_R h(m)) = R = (0_M :_R h(m))$, i.e., $1_R h(m) = 0_M$, a contradiction. Thus we have $(N :_R h(m)) = (N :_R h(M))$, by our hypothesis (3). Since $(N :_R h(m)) = R$, we get $h(M) \subseteq N$.

Proposition 2.4. *Let M be an R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N *is an* S*-weakly prime submodule.*

(ii) $0_M \neq f(K) \subseteq N$ *implies that either* $K \subseteq N$ *or* $f(M) \subseteq N$, *where* $f \in S$ *and* K *is a submodule of* M.

Proof. (1) $\Longrightarrow$ (2) : Let N be an S-weakly prime submodule. Suppose that $f(K) \subseteq N$, $K \nsubseteq N$ and $f(M) \nsubseteq N$. Then we show $f(K) = 0_M$. For every $k \in K$, we have 2 cases:

Case 1: Let $k \in K - N$. By Theorem 2.1, we can see that $(N :_R f(k)) = (N :_R f(M))$ or $(N :_R f(k)) = (0_M :_R f(k))$. Since $f(k) \in f(K) \subseteq N$, one get $1_R \in (N :_R f(k))$. Thus either $1_R \in (N :_R f(M))$ or $1_R \in (0_M :_R f(k))$. The first one contradicts our assumption. Thus we obtain $f(k) = 0_M$.

Case 2: Let $k \in K \cap N$. If $f(k) = 0_M$, we are done. Let suppose $f(k) \neq 0_M$. Since $K \nsubseteq N$, there exists $0_M \neq y \in K - N$. Then $f(y) \in f(K) \subseteq N$, one get $1_R \in (N :_R f(y))$. Thus either $1_R \in (N :_R f(M))$ or $1_R \in (0_M :_R f(y))$. The first one contradicts our assumption. Thus we obtain $f(y) = 0_M$. Then one can see $0_M \neq f(y + k) = f(y) + f(k) \in f(K) \subseteq N$. Since N is S-weakly prime, we get $y + k \in N$ or $f(M) \subseteq N$. So, $y \in N$ or $f(M) \subseteq N$, i.e., contradiction.

Consequently, for every $k \in K$, we obtain $f(k) = 0_M$.

(2) $\Longrightarrow$ (1) : Let $0_M \neq f(m) \in N$. We show that either $m \in N$ or $f(M) \subseteq N$. Since $0_M \neq f(Rm) \subseteq N$, by our hypothesis we obtain either $Rm \subseteq N$ or $f(M) \subseteq N$. Consequently, $m \in N$ or $f(M) \subseteq N$.

Corollary 2.3. *Let M be an R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N *is an* S*-weakly prime submodule.*

(ii) $0_M \neq f(Rm) \subseteq N$ *implies that either* $Rm \subseteq N$ *or* $f(M) \subseteq N$, *where* $f \in S$ *and* $m \in M$.

Proof. By Proposition 2.4.

Theorem 2.2. *Let M be an R-module and N be a proper submodule of M. For all $f \in S$, the followings are equivalent:*

(i) N is an S-prime submodule of M.

(ii) $(N :_R f(x)) = (N :_R f(M))$ for all $x \notin N$.

Proof. $(1) \Longrightarrow (2)$: Assume that N is *S-prime* and $x \notin N$. Let $r \in (N :_R f(x))$. Then we get $rf(x) \in N$. Define $h : M \to M$ such that $h(m) = rf(m)$ for all $m \in M$. Clearly $h \in End(M)$, also $h(x) = rf(x) \in N$. Since N is an *S-prime* submodule and $x \notin N$, then we obtain $h(M) \subseteq N$. Thus $h(M) = rf(M) \subseteq N$ and so $r \in (N :_R f(M))$. The other containment is always hold.

$(2) \Longrightarrow (1)$: Suppose that $h \in End(M)$ and $m \notin N$ such that $h(m) \in N$. We prove that $h(M) \subseteq N$. Since $1_R h(m) \in N$, we get $1_R \in (N :_R h(m)) = (N :_R h(M))$. Thus $h(M) \subseteq N$ by the assumption.

To avoid losing the integrity, we give the following proposition.

Proposition 2.5. *([6], Proposition 2.1) Let M be a faithful cyclic R-module. If N is a weakly prime submodule, then $(N :_R M)$ is a weakly prime ideal of R.*

However, Proposition 2.5 is not true for "S-weakly prime situation". So we mean if N is an S-weakly prime submodule, then $(N :_R M)$ may not be an S-weakly prime ideal of R.

Note that for a subset A of M, we denote the submodule generated by A in M as $< A >$. In particular, if $X = \{a\}$, then it is denoted by $< a >$. If M is an R-module such that $M =< a >$, then M is called cyclic module. It is clear that every cyclic module is a multiplication module, see [13].

Proposition 2.6. *Let M be a cyclic R-module such that $M =< a >$ and N be a proper submodule of M. Then the followings are equivalent:*

(i) N is a weakly prime submodule.

(ii) N is an S-weakly prime submodule.

Proof. $(1) \Longrightarrow (2)$: Assume that N is a weakly prime submodule. Let choose $m \in M$ and $f \in End(M)$ such that $0_M \neq f(m) \in N$ and $m \notin N$. We prove that $f(M) \subseteq N$. Let $f(x) \in f(M)$. Since $M =< a >$, there exist $r_1, r_2 \in R$ such that $x = r_1 a$ and $m = r_2 a$. Then we get $0_M \neq f(m) = f(r_2 a) = r_2 f(a) \in N$. Since N is weakly prime, then $r_2 \in (N :_R M)$ or $f(a) \in N$. If $r_2 \in (N :_R M)$, then we get $m = r_2 a \in N$, i.e., a contradiction. Thus $f(a) \in N$, so $f(x) = r_1 f(a) \in N$. As x is an arbitrary element of M, we obtain $f(M) \subseteq N$.

$(2) \Longrightarrow (1)$: By Proposition 2.1.

Corollary 2.4. *Let M be a faithful cyclic R-module. If N is an S-weakly prime submodule, then $(N :_R M)$ is an S-weakly prime ideal of R.*

Proof. Assume that N is an S-weakly prime submodule. Then N is an weakly prime submodule. By Proposition 2.5, $(N :_R M)$ is a weakly prime ideal of R. Since R is a cyclic R-module, $(N :_R M)$ is an S-weakly prime ideal of R by Proposition 2.6.

For the integrity of our study, we give the following Lemma:

Lemma 2.1. *([13],Corollary in page 231) Let I, J be two ideals of R and M be a finitely generated multiplication R-module. Then $IM \subseteq JM$ if and only if $I \subseteq J + Ann(M)$.*

Theorem 2.3. *Let M be a finitely generated multiplication R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N is an S-prime submodule.

(ii) $(N :_R M)$ is an S-prime ideal of R.

(iii) $N = IM$, for some S-prime ideal I of R with $Ann(M) \subseteq I$.

Proof. $(1) \Longrightarrow (2)$: By Corollary 2.1.5 in [4].

$(2) \Longrightarrow (3)$: Since M is a multiplication module, $N = (N :_R M)M$, so we are done.

$(3) \Longrightarrow (1)$: Suppose that $N = IM$, for some S-prime ideal I of R with $Ann(M) \subseteq I$. To use Proposition 2.2, assume that K is a submodule of M such that $f(K) \subseteq N$, for any $f \in S$. Since M is a multiplication module, there exist two ideals J_1, J_2 of R such that $K = J_1M$ and $f(M) = J_2M$. Then $f(J_1M) = J_1f(M) = J_1J_2M \subseteq N = IM$. By Lemma 2.1, $J_1J_2 \subseteq I + Ann(M) = I$. As I is an S-prime ideal, so prime, we get $J_1 \subseteq I$ or $J_2 \subseteq I$. It implies $J_1M \subseteq N$ or $J_2M \subseteq N$. Consequently, $K \subseteq N$ or $f(M) \subseteq N$.

Theorem 2.4. *Let M be a cyclic faithful R-module and N be a proper submodule of M. Then the followings are equivalent:*

(i) N is an S-weakly prime submodule.

(ii) $(N :_R M)$ is an S-weakly prime ideal of R.

(iii) $N = IM$, for some S-weakly prime ideal I of R.

Proof. $(1) \Longrightarrow (2)$: By Corollary 2.4.

$(2) \Longrightarrow (3)$: Since M is a multiplication module, $N = (N :_R M)M$, so we are done.

$(3) \Longrightarrow (1)$: By the help of Proposition 2.4 and Lemma 2.1, as the previous proof one can prove easily.

Definition 2.2. *([4], Definition 2.1.1)A proper submodule N of an R-module M is said to be fully invariant submodule of M if $f(N) \subseteq N$, for every $f \in S$.*

Theorem 2.5. *Let M be an R-module and N be a fully invariant and S-weakly prime submodule of M that is not S-prime. If I is an ideal of R such that $I \subseteq (N :_R M)$, then $If(N) = 0_M$, for any $f \in S$. In particular, $(N :_R M)f(N) = 0_M$.*

Proof. Suppose that $If(N) \neq 0_M$. We show that N is an S-prime submodule. Let $f(m) \in N$, where $f \in End(M)$ and $m \in M$. If $f(m) \neq 0_M$, since N is S-weakly prime, we are done. So, assume that $f(m) = 0_M$. Then we have 2 cases for $f(N)$.

Case 1: $f(N) \neq 0_M$. Then there exists $n \in N$ such that $f(n) \neq 0_M$. Thus $0_M \neq f(n + m)$. As N is fully invariant, one see $0_M \neq f(n + m) \in N$. Since N is S-weakly prime, $m + n \in N$, i.e., $m \in N$ or $f(M) \subseteq N$.

Case 2: $f(N) = 0_M$. As $If(N) \neq 0_M$, contradiction.

Corollary 2.5. *Let M be an R-module and N be a fully invariant and S-weakly prime submodule of M that is not S-prime. If M is multiplication, then $f(N)^2 = 0_M$, for any $f \in S$.*

Proof. Let M be multiplication. Then $f(N)^2 = (f(N) :_R M)M(f(N) :_R M)M = (f(N) :_R M)(f(N) :_R M)M = (f(N) :_R M)f(N) \subseteq (N :_R M)f(N)$, since N is fully invariant. By Theorem 2.5, $(N :_R M)f(N) = 0_M$, so $f(N)^2 = 0_M$.

Corollary 2.6. *Let M be a multiplication R-module and N, K be fully invariant and S-weakly prime submodules of M that are not S-prime. Then $f(N)f(K) \subseteq \sqrt{0_M}$.*

Proof. Assume that $f(b) \in f(K)$. Then $f(b)^2 = Rf(b)Rf(b) \subseteq f(K)^2 = 0_M$, by Corollary 2.5. Then we get $f(K) \subseteq \sqrt{0_M}$. Similarly, $f(N) \subseteq \sqrt{0_M}$. Then we obtain $f(N)f(K) \subseteq \sqrt{0_M}$.

For the next proof, we will need the following Lemma:

Lemma 2.2. *([6], Lemma 2.5) Let M be a multiplication module over R. Let N and K be submodules of M. Then the followings are hold:*

(i) If for every $a \in N, aK = 0_M$, then $NK = 0_M$.

(ii) If for every $b \in K, Nb = 0_M$, then $NK = 0_M$.

(iii) If for every $a \in N, b \in K, ab = 0_M$, then $NK = 0_M$.

Theorem 2.6. *Let M be a finitely generated faithful multiplication R-module and N be a fully invariant and S-weakly prime submodule of M that is not S-prime. If $f \in S$ is onto, then $f(N)f(\sqrt{0_M}) = 0_M$.*

Proof. Let $y = f(x) \in f(\sqrt{0_M})$ such that $x \in \sqrt{0_M}$. Then there exists two ideals I, J in R such that $f(N) = IM$ and $Rx = JM$. Then as f is onto, one see that $Rf(x) = f(Rx) = f(JM) = Jf(M) = JM$. For $f(x)$, we have 2 cases :

Case 1: Let $f(x) \in f(N)$. Then $Rf(x) \subseteq f(N)$. By Lemma 2.1, we get $J \subseteq I$. Thus with Corollary 2.5, we have $f(N)Rf(x) = IJM \subseteq f(N)^2 = 0_M$. By Lemma 2.2, $f(N)f(\sqrt{0_M}) = 0_M$.

Case 2: Let $f(x) \notin f(N)$. Then we get $x \notin N$. By Theorem 2.1, $(N :_R f(x)) = (0_M :_R f(x))$ or $(N :_R f(x)) = (N :_R f(M))$.

Assume that $(N :_R f(x)) = (0_M :_R f(x))$. Thus $(N :_R M)M \subseteq (N :_R f(x))M = (0_M :_R f(x))M$, so, as N is fully invariant, $f(N) \subseteq (0_M :_R f(x))M$. Then $f(N)Rf(x) \subseteq (0_M :_R f(x))Rf(x) = 0_M$, i.e., $f(N)f(x) = 0_M$. By Lemma 2.2, $f(N)f(\sqrt{0_M}) = 0_M$.

Now, suppose that $(N :_R f(x)) = (N :_R f(M))$. As $x \in \sqrt{0_M}$, there exists a smallest positive integer n such that $x^n = 0_M$ and $x^{n-1} \neq 0_M$. Then we see $Rx^n = J^nM = 0_M$ and $Rx^{n-1} = J^{n-1}M \neq 0_M$. Hence, since $Rf(x) = JM$, one get $Rf(x)^n = J^nM = 0_M$ and $Rf(x)^{n-1} = J^{n-1}M \neq 0_M$. Moreover, we have $J^nM = 0_M$ implies $J^n = 0_R$, by Lemma 2.1. Then it is clear that $J^{n-1} \subseteq (I :_R J)$. Hence, as N is fully invariant, $0_M \neq Rf(x)^{n-1} = J^{n-1}M \subseteq (IM :_R JM)M = (f(N) :_R Rf(x))M \subseteq (N :_R Rf(x))M$. Then by our hypothesis and as f is onto, $0_M \neq Rf(x)^{n-1} \subseteq (N :_R Rf(x))M = (N :_R f(M))M = (N :_R M)M = N$, this implies $0_M \neq f(x^{n-1}) \subseteq N$. Since N is S-weakly prime, we get $0_M \neq x^{n-1} \subseteq N$ or $f(M) \subseteq N$. The second one contradicts with $f(x) \notin N$. As every S-weakly prime is a weakly prime submodule and by Theorem 2.6 in [6], $0_M \neq Rx^{n-1} \subseteq N$ implies $Rx \subseteq N$, so $f(x) \in f(N)$, a contradiction.

Corollary 2.7. *Let M be a finitely generated faithful multiplication R-module and N, K be fully invariant and S-weakly prime submodules of M that are not prime. If $f \in S$ is onto, then $f(N)f(K) = 0_M$.*

Proof. Assume that $f(b) \in f(K)$. Then $f(b)^2 \subseteq f(K)^2 = 0_M$, by Corollary 2.5. Then we get $f(K) \subseteq \sqrt{0_M}$. As N is fully invariant, one see $f(N)f(K) \subseteq f(N)\sqrt{0_M} \subseteq N\sqrt{0_M}$. Since N is S-weakly prime (so weakly prime) and not prime, we know $N\sqrt{0_M} = 0_M$, by the help of Theorem 2.7 in [6]. Consequently, $f(N)f(K) = 0_M$.

Corollary 2.8. *Let M be a finitely generated faithful multiplication module over R with unique maximal submodule K and every prime of M is maximal. Let N be a fully invariant and S-weakly prime submodule of M. If $f \in S$ is onto, then $N = K$ or $f(N)f(K) = 0_M$.*

Proof. If N is S-prime, so prime, by our hypothesis $N = K$. If N is not S-prime, one see $\sqrt{0_M} = \bigcap_{N_i \in Spec(M)} N_i = K$, where $Spec(M)$ is the set of all prime submodules of M. Then we obtain $f(\sqrt{0_M}) = f(K)$. Thus $f(N)f(K) = f(N)f(\sqrt{0_M}) = 0_M$, by Theorem 2.6.

Corollary 2.9. *Let M be a finitely generated faithful module over a local ring R with unique maximal submodule K and every prime of M is maximal. Let N be a fully invariant and S-weakly prime submodule of M. If $f \in S$ is onto, then $N = K$ or $f(N)f(K) = 0_M$.*

Proof. By Corollary 1 in [10], M is cyclic. Thus M is multiplication R-module. By Corollary 2.8, it is done.

Let M_1, M_2 be R-modules and we know that $M = M_1 \times M_2$ is an R-module. For every $f_i \in End(M_i)$, let define $f : M \to M$ with $f((m_1, m_2)) = (f_1(m_1), f_2(m_2))$, for every $(m_1, m_2) \in M$, $i = 1, 2$. Then one can easily see, $f \in End(M)$ and $f(M) = f_1(M_1) \times f_2(M_2)$. Also, we use the following notations: $S_1 = End(M_1)$, $S_2 = End(M_2)$ and $S = S_1 \times S_2$.

Theorem 2.7. *Let M_1, M_2 be R-modules and N_1 be a proper submodule of M_1. Then the followings are equivalent:*

(i) $N = N_1 \times M_2$ *is an S-prime submodule of* $M = M_1 \times M_2$.

(ii) N_1 *is an S_1-prime submodule of* M_1.

Proof. $(1) \Longrightarrow (2)$: Suppose that $N = N_1 \times M_2$ is an S-prime submodule of $M = M_1 \times M_2$. Let $f_1(m_1) \in N_1$, for some $m_1 \in M_1$ and $f_1 \in End(M_1)$. Then for every $m_2 \in M_2$ and $f_2 \in End(M_2)$, we get $f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N_1 \times M_2 = N$. So, as $N = N_1 \times M_2$ is an S-prime submodule of $M = M_1 \times M_2$, we get $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$. Thus, by $f(M) = f_1(M_1) \times f_2(M_2)$, one see $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$, i.e., N_1 is S_1-prime.

$(2) \Longrightarrow (1)$: Let N_1 be an S_1-prime submodule of M_1. Assume that $f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N = N_1 \times M_2$, for some $(m_1, m_2) \in M, f_1 \in End(M_1)$ and $f_2 \in End(M_2)$. Then $f_1(m_1) \in N_1$. As N_1 is S_1-prime, we have $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$. Thus $(m_1, m_2) \in N$ or $f(M_1 \times M_2) = f_1(M_1) \times f_2(M_2) \subseteq N = N_1 \times M_2$.

Theorem 2.8. *Let M_1, M_2 be R-modules and N_1 be a proper submodule of M_1. Then the followings are equivalent:*

(i) $N = N_1 \times M_2$ *is an S-weakly prime submodule of* $M = M_1 \times M_2$.

(ii) N_1 *is S_1-weakly prime and for every* $(m_1, m_2) \in M$, $f_1 \in S_1$ *and* $f_2 \in S_2$, *if* $f_1(m_1) = 0_{M_1}, m_1 \notin N_1$ *and* $f_1(M_1) \nsubseteq N_1$, *then* $f_2(m_2) = 0_{M_2}$.

Proof. (1) $\Longrightarrow$ (2) : Assume that $N = N_1 \times M_1$ is an S-weakly prime submodule of $M = M_1 \times M_2$. Firstly, we show that N_1 is S_1-weakly prime. Let $0_{M_1} \neq f_1(m_1) \in N_1$, for some $m_1 \in M_1$. Then for every $m_2 \in M_2$, we get $0_M \neq f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N$. So, as $N = N_1 \times M_2$ is an S-weakly prime submodule of $M = M_1 \times M_2$, we get $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$. Thus $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$, i.e., N_1 is S_1-weakly prime. Let $0_{M_1} = f_1(m_1) \in N_1$ such that $m_1 \notin N_1$ and $f_1(M_1) \nsubseteq N_1$. Then for every $m_2 \in M_2$, we say $(m_1, m_2) \notin N$ and $f(M_1 \times M_2) \nsubseteq N$. Moreover, if $f_2(m_2) \neq 0_{M_2}$, we have $0_M \neq f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N$, so $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$, a contradiction. Consequently, $f_2(m_2) = 0_{M_2}$.

(2) $\Longrightarrow$ (1) : Suppose that the condition (2) is true.

Let $0_M \neq f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N$, for some $(m_1, m_2) \in M$. Then for $f_1(m_1)$, we have 2 cases:

Case 1: Let $0_{M_1} \neq f_1(m_1)$. Since $f(m_1) \in N_1$ and N_1 is S_1-weakly prime, we get $m_1 \in N_1$ or $f(M_1) \subseteq N_1$, i.e., $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$. Thus, it is done.

Case 2: Let $0_{M_1} = f_1(m_1)$. Then $0_{M_2} \neq f_2(m_2)$. Assume that $m_1 \notin N_1$ and $f_1(M_1) \nsubseteq N_1$. Then by (2), $f_2(m_2) = 0_{M_2}$, a contradiction. So $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$. Thus $(m_1, m_2) \in N$ or $f(M) \subseteq N$.

Proposition 2.7. *Let M_1, M_2 be R-modules and N_1, N_2 be a proper submodule of M_1, M_2, resceptively. If $N = N_1 \times N_2$ is an S-prime submodule of $M = M_1 \times M_2$, then N_1 is an S_1-prime submodule of M_1 and N_2 is an S_2-prime submodule of M_2.*

Proof. Suppose that $N = N_1 \times N_2$ is an S-prime submodule of $M = M_1 \times M_2$. Now, we will show that N_1 is an S_1-prime submodule. Take $f_1 \in S_1$ such that $f_1(m_1) \in N_1$ for some $m_1 \in M_1$. Also take $f_2 \in S_2$. Then $f((m_1, 0_{M_2})) = (f_1(m_1), f_2(0_{M_2})) = (f_1(m_1), 0_{M_2}) \in N_1 \times N_2 = N$. Since N is S-prime, $(m_1, 0_{M_2}) \in N$ or $f(M_1 \times M_2) \subseteq N$. This implies that $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$. Similar argument shows that N_2 is an S_2-prime submodule.

Proposition 2.8. *Let M_1, M_2 be R-modules and N_1, N_2 be a proper submodule of M_1, M_2, resceptively. If $N = N_1 \times N_2$ is an S-weakly prime submodule of $M = M_1 \times M_2$, then N_1 is an S_1-weakly prime submodule of M_1 and N_2 is an S_2-weakly prime submodule of M_2.*

Proof. Assume that $N = N_1 \times N_2$ is an S-weakly prime submodule of $M = M_1 \times M_2$. Let $f_1 \in S_1$ such that $0_{M_1} \neq f_1(m_1) \in N_1$ for some $m_1 \in M_1$. Take $f_2 \in S_2$. Then $0_M \neq f((m_1, 0_{M_2})) \in N$. Since N is S-weakly prime, we get either $(m_1, 0_{M_2}) \in N$ or $f(M_1 \times M_2) \subseteq N$. This yields that $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$. Similarly, one can see that N_2 is S_2-weakly prime.

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$, for some submodules N_1 of M_1 and N_2 of M_2, see [12].

Theorem 2.9. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times M_2$ is a proper submodule of M. Then the followings are equivalent:*

(i) N_1 is an S_1-prime submodule of M_1.

(ii) N is an S-prime submodule of M.

(iii) N is an S-weakly prime submodule of M.

Proof. (1) $\Longrightarrow$ (2) : Let N_1 be an S_1-prime submodule of M_1.

Assume that $f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N = N_1 \times M_2$ for some $(m_1, m_2) \in M$. Then $f_1(m_1) \in N_1$. As N_1 is S_1-prime, we have $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$. Thus $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$.

(2) $\Longrightarrow$ (3) : It is clear.

(3) $\Longrightarrow$ (1) : Suppose that N is an S-weakly prime submodule of M. Let $0_{M_1} \neq f_1(m_1) \in N_1$ for some $m_1 \in M_1$. Put $f_2 = id_{M_2}$, where id_{M_2} denotes the identity homomorphism of M_2. Then for every $m_2 \in M_2$, we get $f((m_1, m_2)) = (f_1(m_1), f_2(m_2)) \in N_1 \times M_2 = N$. As $0_{M_1} \neq f_1(m_1)$, we get $0_M \neq (f_1(m_1), f_2(m_2))$. By our hypothesis, $(m_1, m_2) \in N$ or $f(M_1 \times M_2) \subseteq N$. Consequently, $m_1 \in N_1$ or $f_1(M_1) \subseteq N_1$.

Theorem 2.10. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that N_1, N_2 be a proper submodule of M_1, M_2, resceptively. Then the followings are hold:*

(i) If $N = N_1 \times N_2$ is an S-prime submodule of $M = M_1 \times M_2$, then N_1 is an S_1-prime submodule of M_1 and N_2 is an S_2-prime submodule of M_2.

(ii) If $N = N_1 \times N_2$ is an S-weakly prime submodule of $M = M_1 \times M_2$, then N_1 is an S_1-weakly prime submodule of M_1 and N_2 is an S_2-weakly prime submodule of M_2.

Proof. (1) : It can be easily proved similar to Proposition 2.7.

(2) : Similar to Proposition 2.8.

References

[1] Ameri, R. (2003). On the prime submodules of multiplication modules. Inter. J. of Math. and Mathematical Sciences, 27: 1715–1724.

[2] Anderson, D. D., Smith, E. (2003). Weakly prime ideals. Houston J. of Math., 29: 831–840.

[3] Atiyah, M. F., MacDonald, I. G. (1969). Introduction to commutative algebra. CRC Press.

[4] Dakheel, S. O. (2010). S-prime submodules and some related concepts (Thesis).

[5] Dauns, J. (1978). Prime modules. J. reine Angew. Math., 2: 156–181.

[6] Ebrahimi, S., Farzalipour, F. (2007). On weakly prime submodules. Tamkang journal of Mathematics, 38(3): 247-252.

[7] El-Bast, Z. A., Smith, P. F. (1988) Multiplication modules. Comm. in Algebra, 16: 755-779.

[8] Gungoroglu, G. (2000). Strongly Prime Ideals in CS-Rings. Turk. J. Math., 24: 233-238.

[9] Lu, C. P. (1984). Prime submodules of modules. Comm. Math. Univ. Sancti Pauli, 33: 61–69.

[10] Lu , C. P. (1995). Spectra of modules, Comm. in Algebra, 23: 3741-3752.

[11] McCasland R. L., Moore, M. E. (1992). Prime submodules. Comm. Algebra, 20: 1803–1817.

[12] Mostafanasab H., Tekir, U., Oral, K. H. (2016). Weakly classical prime submodules. Kyunhpook Math. J., 56: 10085-1101.

[13] Smith, P. F. (1988). Some remarks on Multiplication modules. Arch Math, 50: 223-235.