EURASIAN BULLETIN OF MATHEMATICS
EBM (2021), VOL. 4, NO. 2, 98-106

On m^*-g-closed sets and m^*-R_0 spaces in a hereditary m-*space* (X, m, H)

Ahu Acikgoz[1,*], Takashi Noiri[2]
Department of Mathematics, Balikesir University, 10145 Turkey[1]
2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken,869-5142 Japan[2]

Abstract. The minimal local function and the minimal structure m^*_H which includes m in a hereditary minimal space (X, m, H) have been described by Noiri and Popa [22]. Noiri and Popa [22] also have introduced and investigated the notion of $m - H_g-$closed sets and $(\Lambda, m^*_H)-$ closed sets in a hereditary minimal space (X, m, H). We describe the notions of $m^* - g-$closed sets and $m^* - H_g-$ closed sets in a hereditary minimal space (X, m, H) and study at some of their fundamental features and characterizations in this study.

2010 Mathematics Subject Classifications: 54A05, 54A10, 54D10
Key Words and Phrases: $m^* - H_g - closed$; $m^* - g-$closed; $m^* - R_1$; $m - R_0$; $m^* - R_0$

Popa and Noiri [28] were the first to explore the concept of minimal spaces. If $\emptyset \in m$ and $X \in m$, a subfamily m of a nonempty set X is termed a minimal structure.Since then, there has been a growing tendency among topologists to investigate this idea. As a result, a large number of surveys (e.g. [1-7, 9, 11, 14-18, 20, 23-27, 29-33]) were conducted as a result of this notion.

Kuratowski [12] introduced the concept of ideals in topological spaces. Janković and Hamlett [10] introduced the local function in an ideal topological space (X, τ, I) and subsequently created the topology τ^*. And they examined in detail the properties of this topology.

In an ideal minimal space, Ozbakır and Yildirim [23] described and researched the ideas of m^*-closed sets and $m - I_g-$closed sets. The $m-$local function and minimal $^*-$closures in an ideal minimal space (X, m, I) are presented and studied. At the same time they established the minimal structure m^* providing that $m^* \supset m$. Besides that the concept of $m - I_g-$closed sets is described and researched in [23].

T. Noiri and V. Popa [22] have described a new set–operator on minimal spaces based on Császár [8]'s hereditary class. A hereditary minimal space (also known as hereditary m-space) is a minimal space (X, m) with a hereditary class H on X and is denoted by (X, m, H).In a hereditary minimal space (X, m, H), Noiri and Popa [22] have presented $m - H_g$-closed sets and $(\Lambda, m^*_H)-$closed sets. They have also used $m - H_g$-closed sets and $(\Lambda, m^*_H)-$closed sets to produce decompositions of m^*_H-closed sets.

We review the basic principles needed for the research in the first part of this study. We present the notion of $m^* - g-$closed sets and $m^* - H_g-$closed sets in a hereditary minimal space (X, m, H) in the second part. We investigate their features and demonstrate that an $m^* - g-$closed set is weaker than a $m-$closed set but stronger than a $mg-$closed set. In the last part, we present several new types of separation axioms termed $m^* - R_0$ and $m^* - R_1$ and study some of their characterizations using $m^* - g-$closed sets.

*Corresponding author.

Email address: `ahuacikgoz@gmail.com`* (Corresponding author)

1. Preliminaries

Definition 1.1. *A subfamily m of the power set $P(X)$ of a nonempty set X is called a minimal structure (briefly $m-$structure) [28] on X if $\emptyset \in m$ and $X \in m$.*

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an m-space. Each member of m is said to be $m-$open and the complement of an $m-$open set is said to be $m-$closed. For a point $x \in X$, the family $\{U : x \in U$ and $U \in m\}$ is denoted by $m(x)$.

Definition 1.2. *Let (X, m) be an $m-$space and A a subset of X. The $m-$closure $mCl(A)$ of A [13] is described by $mCl(A) = \cap\{F \subset X : A \subset F,\ X\backslash F \in m\}$.*

Lemma 1.3. *(Maki et al. [13]). Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $A \subset mCl(A)$ and $mCl(A) = A$ if A is $m-$closed,

(2) $mCl(\emptyset) = \emptyset$, $mCl(X) = X$,

(3) If $A \subset B$, then $mCl(A) \subset mCl(B)$,

(4) $mCl(A) \cup mCl(B) \subset mCl(A \cup B)$,

(5) $mCl(mCl(A)) = mCl(A)$.

Definition 1.4. *A minimal structure m of a set X is said to have property ß [13] if the union of any collection of elements of m is an element of m.*

Lemma 1.5. *(Popa and Noiri [28]). Let (X, m) be an $m-$space and A a subset of X.*

(1) *$x \in mCl(A)$ if and only if $U \cap A \neq \varnothing$ for every $U \in m(x)$.*

(2) *Let m have property ß. Then the following properties hold:*

(i) *A is $m-$closed if and only if $mCl(A) = A$,*

(ii) *$mCl(A)$ is $m-$closed.*

Definition 1.6. *A nonempty subfamily H of $P(X)$ is called a hereditary class on X [8] if it satisfies the following properties: $A \in H$ and $B \subset A$ implies $B \in H$. A hereditary class H is called an ideal if it satisfies the additional condition: $A \in H$ and $B \in H$ implies $A \cup B \in H$.*

A minimal space (X, m) with a hereditary class H on X is called a hereditary minimal space (briefly hereditary $m-$space) [22] and is denoted by (X, m, H).

Definition 1.7. *Let (X, m, H) be a hereditary $m-$space. For a subset A of X, the minimal local function $A^*_{mH}(H, m)$ of A [22] is described as follows:*

$$A^*_{mH}(H, m) = \{x \in X : U \cap A \notin H \ for\ every\ U \in m(x).$$

Hereafter, $A^*_{mH}(H, m)$ is simply denoted by A^*_{mH}.

Remark 1.8. *(Noiri and Popa [22]). Let (X, m, H) be a hereditary $m-$space and A a subset of X.*

(1) If $H = \{\emptyset\}$ (resp. $P(X)$), then $A^*_{mH} = mCl(A)$ (resp. $A^*_{mH} = \emptyset$).

(2) If $A \in H$, then $A^*_{mH} = \emptyset$.

Lemma 1.9. *(Noiri and Popa [22]). Let (X, m, H) be a hereditary $m-$space. For subsets A and B of X, the following properties hold:*

(1) If $A \subset B$, then $A^*_{mH} \subset B^*_{mH}$,

(2) $A^*_{mH} = mCl(A^*_{mH}) \subset mCl(A)$,

(3) $A^*_{mH} \cup B^*_{mH} \subset (A \cup B)^*_{mH}$,

(4) $(A^*_{mH})^*_{mH} \subset (A \cup A^*_{mH})^*_{mH} = A^*_{mH}$.

Proposition 1.10. *(Noiri and Popa [22]). For a hereditary $m-$space (X, m, H), $X = X^*_{mH}$ if and only if $m \cap H = \{\emptyset\}$.*

Definition 1.11. *(Noiri and Popa [22]). Let (X, m, H) be a hereditary $m-$space and A a subset of X. The minimal $* -$ closure $mCl^*_H(A)$ of A is described as $mCl^*_H(A) = A \cup A^*_{mH}$. The m^*_H-structure is defined as follows:*

$$m^*_H = \{U \subset X : mCl^*_H(X \backslash U) = X \backslash U\}.$$

Each member of m^*_H is said to be $m^*_H -$ open and the complement of an m^*_H $-$open set is said to be m^*_H $-$closed.

Definition 1.12. *(Noiri and Popa [22]). Let (X, m, H) be a hereditary $m-$space. A subset A of X is said to be $m - H_g -$ closed (resp. $mg -$ closed [19]) if $A^*_{mH} \subset U$ (resp. $mCl(A) \subset U$) whenever $A \subset U$ and $U \in m$. A subset A of X is said to be $m - H_g -$ open if the complement of A is $m - H_g -$ closed.*

Lemma 1.13. *(Noiri and Popa [22]). Let (X, m, H) be a hereditary m-space and A, B subsets of X. Then, the following properties hold:*

(1) $A \subset mCl^*_H(A)$,
(2) $mCl^*_H\ (\emptyset) = \emptyset$ *and* $mCl^*_H(X) = X$,
(3) *If* $A \subset B$, *then* $mCl^*_H\ (A) \subset mCl^*_H(B)$,
(4) $mCl^*_H\ (A) \cup mCl^*_H\ (B) \subset mCl^*_H\ (A \cup B)$,
(5) $mCl^*_H(mCl^*_H\ (A)) = mCl^*_H\ (A)$, *that is,* $mCl^*_H\ (A)$ *is* $m^*_H -$ *closed.*

2. m^*-g-$Closed$ sets

In this section, we describe the notions of $m^* - g - closed$ sets and $m^* - H_g - closed$ sets and study their basic properties and several characterizations.

Definition 2.1. *A subset A of a hereditary $m-$space (X, m, H) is said to be $m^* - H_g -$closed (resp. $m^* - g -$ closed) set if $A^*_{mH} \subset U$ (resp. $mCl(A) \subset U$) whenever $A \subset U$ and U is $m^*_H -$ open. A subset A of X is said to be $m^* - H_g -$open (resp. $m^* - g -$open) if its complement is $m^* - H_g -$closed (resp. $m^* - g -$ closed).*

Proposition 2.2. *Let (X, m, H) be a hereditary $m-$space. Then for a subset of X, the following implications hold:*

$$\begin{array}{ccc} m - closed & \Rightarrow & m^*_H - closed \\ \Downarrow & & \Downarrow \\ m^* - g - closed & \Rightarrow & m^* - H_g - closed \\ \Downarrow & & \Downarrow \\ mg - closed & \Rightarrow & m - H_g - closed \end{array}$$

Proof. By [21], every $m-$closed set is m^*_H-closed and every $m-$closed set is $mg-$closed. It is seen that every m^*_H-closed set is $m^* - H_g - closed$. Since $m \subseteq m^*_H$, every $m-$open set is m^*_H $-$open and it follows that every $m^* - H_g-$closed set is $m - H_g-$closed and every $m^* - g-$closed set is $mg-$closed in a hereditary minimal space (X, m, H).

Theorem 2.3. *Let (X, m, H) be a hereditary m-space and $A \subset X$. Then A is $m^* - g-$closed if and only if for every $x \in mCl(A)$, $mCl^*_H(\{x\}) \cap A \neq \varnothing$.*

Proof. Necessity. Let A be $m^* - g-$closed and $x \in mCl(A)$. Suppose $mCl^*_H(\{x\}) \cap A = \varnothing$. Then $A \subseteq X \backslash mCl^*_H(\{x\})$, where $X \backslash mCl^*_H(\{x\})$ is m^*_H-open. Since A is $m^* - g-$closed, it follows that $mCl(A) \subseteq X \backslash mCl^*_H(\{x\})$ and thus $mCl(A) \cap mCl^*_H(\{x\}) = \varnothing$. This is a contradiction.

Sufficiency. Let $A \subseteq U$ and U be m^*_H-open. Suppose that $mCl(A) \nsubseteq U$. Then there exists $x \in mCl(A)$ providing that $x \notin U$. Thus $x \in X \backslash U$ and $mCl^*_H(\{x\}) \subseteq X \backslash U \subseteq X \backslash A$. Therefore $mCl^*_H(\{x\}) \cap U = \varnothing$ and $mCl^*_H(\{x\}) \cap A = \varnothing$. This is a contradiction. So $mCl(A) \subseteq U$ and consequently A is $m^* - g-$closed.

Theorem 2.4. *A subset A of a hereditary $m-$space (X, m, H) is $m^* - g-$open if and only if $F \subset mInt(A)$ whenever $F \subset A$ and F is m^*_H-closed, where $mInt(A)$ is described as follows: $mInt(A) = \cup\{U : U \in m, U \subset A\}$ [13].*

Proof. Necessity. Assume that A is $m^* - g-$open. Let $F \subset A$ and F be m^*_H-closed. Then $X \backslash A \subset X \backslash F \in m^*_H$ and $X \backslash A$ is $m^* - g-$closed. Therefore, we have $mCl(X \backslash A) \subset X \backslash F$ and $X \backslash mInt(A) = mCl(X \backslash A) \subset X \backslash F$ and thus $F \subset mInt(A)$.

Sufficiency. Let $X \backslash A \subset G$ and G be m^*_H-open. Then $X \backslash G \subset A$ and $X \backslash G$ is m^*_H-closed. Now by given condition, we have $X \backslash G \subset mInt(A)$ and thus $mCl(X \backslash A) \subset G$. Therefore, $X \backslash A$ is $m^* - g-$closed and A is $m^* - g-$open.

Theorem 2.5. *Let (X, m, H) be a hereditary $m-$space and m have property ß. For a subset A of X, the following properties are equivalent:*

(1) *A is an $m^* - g-$closed set.*

(2) *$mCl(A) \backslash A$ contains no non$-$empty m^*_H-closed set.*

(3) *$mCl(A) \backslash A$ is $m^* - g-$open.*

Proof. (1) $\Rightarrow$ (2) : Assume that A is $m^* - g-$closed. Let $F \subseteq mCl(A) \backslash A$, where F is an m^*_H-closed set.Then $X \backslash (mCl(A) \backslash A) \subseteq X \backslash F$ and $(X \backslash mCl(A)) \cup A \subseteq X \backslash F$. Since $A \subseteq X \backslash F \in m^*_H$, $mCl(A) \subseteq X \backslash F$ (since A is $m^* - g-$closed). Therefore, $F \subseteq (X \backslash mCl(A)) \cap mCl(A) = \varnothing$.

(2) $\Rightarrow$ (3) : Let $F \subseteq mCl(A) \backslash A$, where F is an m^*_H-closed set. It follows from (2) that $F = \varnothing$; thus $F \subseteq mInt(mCl(A) \backslash A)$. Therefore, by Theorem 2.4, $mCl(A) \backslash A$ is $m^* - g-$open.

(3) $\Rightarrow$ (1) : Assume that $A \subseteq U$, where U is m^*_H-open. Then $mCl(A) \cap (X \backslash U) \subseteq mCl(A) \cap (X \backslash A) = mCl(A) \backslash A$ and by (3) $mCl(A) \backslash A$ is $m^* - g-$open. Since $mCl(A)$ is $m-$closed by Lemma 1.5 and every $m-$closed set is m^*_H-closed, $mCl(A) \cap (X \backslash U)$ is m^*_H-closed and $mCl(A) \backslash A$ is $m^* - g-$open, by Theorem 2.4, we have $mCl(A) \cap (X \backslash U) \subseteq mInt(mCl(A) \backslash A) = \varnothing$. Therefore $mCl(A) \subseteq U$. This shows that A is $m^* - g-$closed.

Theorem 2.6. *Let (X, m, H) be a hereditary m-space and A, B subsets of X. If $A \subseteq B \subseteq mCl(A)$ and A is $m^* - g-$closed, then B is also an $m^* - g-$closed set.*

Proof. Let $B \subseteq U$, where U is m^*_H-open. Then $A \subseteq U$. Since A is $m^* - g-$closed, $mCl(A) \subseteq U$ and $mCl(B) \subseteq U$. Therefore, B is an $m^* - g-$closed set.

Theorem 2.7. *Let (X, m, H) be a hereditary m-space and A, B subsets of X. If $mInt(A) \subseteq B \subseteq A$ and A is $m^* - g-$open, then B is also $m^* - g-$open.*

Proof. The proof follows from Theorem 2.6.

Theorem 2.8. *Let (X, m, H) be a hereditary $m-$space and m have property ß. Then an $m^* - g-$closed set A is $m-$closed if and only if $mCl(A) \backslash A$ is m^*_H-closed.*

Proof. Necessity. If an m^*-g-closed set A is $m-$closed, then it is obvious that $mCl(A)\backslash A(=\varnothing)$ is m_H^*-closed.

Sufficiency. Let A be an m^*-g-closed set providing that $mCl(A)\backslash A$ is m_H^*- closed. Since A is m^*-g-closed, so by Theorem 2.5, $mCl(A)\backslash A=\varnothing$. Therefore, A is $m-$closed.

Theorem 2.9. *Let (X,m,H) be a hereditary $m-$space and m have property ß. Then the following are equivalent:*

(1) *m_H^*-open sets are $m-$closed.*

(2) *Any subset of X is m^*-g-closed.*

Proof. $(1)\Rightarrow(2)$: Let $A\subseteq U$ and U be m_H^*-open. Then from (a), U is $m-$closed. Therefore $mCl(A)\subseteq mCl(U)=U$ and A is m^*-g-closed.

$(2)\Rightarrow(1)$: Let U be an m_H^*-open set. Then from (2), U is m^*-g-closed and $mCl(U)\subseteq U$. Thus U is $m-$closed.

Corollary 2.10. *Let (X,m,H) be a hereditary $m-$space and m have property ß. Then every m_H^*-open and m^*-g-closed set is $m-$closed.*

Theorem 2.11. *Let (X,m,H) be a hereditary $m-$space and m have property ß. A subset A of X is m^*-g-closed if and only if $A=F\backslash K$, where F is $m-$closed and K contains no nonempty m_H^*-closed set.*

Proof. Necessity. Let A be an m^*-g-closed set. Then by Theorem 2.5 $mCl(A)\backslash A=K$ (put) contains no nonempty m_H^*-closed set. Put $F=mCl(A)$. Therefore $F\backslash K=mCl(A)\backslash(mCl(A)\backslash A)=A$.

Sufficiency. Suppose $A=F\backslash K$ providing that F is $m-$closed and K contains no nonempty m_H^*-closed set. Let $A\subseteq U$, where U is m_H^*-open. From $F\backslash K\subseteq U$ we obtain $F\cap(X\backslash U)\subseteq K$. Since $F\cap(X\backslash U)$ is an m_H^*-closed set, we get $F\cap(X\backslash U)=\varnothing$ and $F\subseteq U$. Since $A\subseteq F$, $mCl(A)\subseteq F\subseteq U$. Thus A is m^*-g-closed.

Theorem 2.12. *Let (X,m,H) be a hereditary $m-$space and m have property ß. Then $A\subseteq X$ is m^*-g-open if and only if $U=X$ whenever U is m_H^*-open and $mInt(A)\cup(X\backslash A)\subseteq U$.*

Proof. Necessity. Let A be m^*-g-open and U be an m_H^*-open set providing that $mInt(A)\cup(X\backslash A)\subseteq U$. In this case, $X\backslash U\subseteq X\backslash(mInt(A)\cup(X\backslash A))=mCl(X\backslash A)\backslash(X\backslash A)$. Because $(X\backslash A)$ is m^*-g-closed, it is seen by Theorem 2.5, that $(X\backslash U)=\varnothing$ and thus $U=X$.

Sufficiency. Let $F\subseteq A$ and F be m_H^*-closed. Then $mInt(A)\cup(X\backslash A)\subseteq mInt(A)\cup(X\backslash F)$. We take attention that $mInt(A)\cup(X\backslash F)$ is m_H^*-open. Then under given situations, $mInt(A)\cup(X\backslash F)=X$ and $F\subseteq mInt(A)$. Therefore A is an m^*-g-open set (by Theorem 2.4).

3. m^*-R_0 Spaces

In this part, we use m^*-g-closed sets to investigate new types of separation axioms in a hereditary $m-$space (X,m,H).

Definition 3.1. *An $m-$space (X,m) is said to be $m-R_0$ [7] if for each $m-$open set U and each $x\in U$, $mCl(\{x\})\subseteq U$.*

The notion of m^*-R_0 spaces is described as follows:

Definition 3.2. *A hereditary $m-$space (X,m,H) is said to be m^*-R_0 if for every m_H^*-open set U and each $x\in U$, $mCl(\{x\})\subseteq U$.*

Remark 3.3. *Since $m-$open sets are m^*_H-open, every m^*-R_0 space is $m-R_0$.*

Theorem 3.4. *Let (X, m, H) be a hereditary m-space. Then the following statements are equivalent:*

(1) A hereditary $m-$space (X, m, H) is m^*-R_0.

(2) $x \in mCl^*_H(\{y\})$ if and only if $y \in mCl(\{x\})$ for any two distinct points x and y.

Proof. (1) $\Rightarrow$ (2): Let $x \in mCl^*_H(\{y\})$ and U be an $m-$open set containing y. Then by (1), $mCl(\{y\}) \subseteq U$. Since $mCl^*_H(\{y\}) \subseteq mCl(\{y\}) \subseteq U$, $x \in U$ and thus $y \in mCl(\{x\})$. Let $y \in mCl(\{x\})$ and U be any m^*_H-open set containing x. Then using (1), we get $mCl(\{x\}) \subseteq U$ and $y \in U$. This shows that $x \in mCl^*_H(\{y\})$.

(2) $\Rightarrow$ (1) : Suppose U is any m^*_H-open set and $x \in U$. If $y \notin U$, then $x \notin mCl^*_H(\{y\})$ and thus $y \notin mCl(\{x\})$. It is seen that $mCl(\{x\}) \subseteq U$. Therefore, the hereditary $m-$space (X, m, H) is m^*-R_0.

In the following theorem, we obtain some characterizations of m^*-R_0 property.

Theorem 3.5. *For a hereditary $m-$space (X, m, H), where m has property ß, the following properties are equivalent:*

(1) A hereditary $m-$space (X, m, H) is m^*-R_0.

(2) If F is an m^*_H-closed set and $x \notin F$, then there exists an $m-$open set U providing that $x \notin U$ and $F \subseteq U$.

(3) If F is an m^*_H-closed set and $x \notin F$, then $F \cap mCl(\{x\}) = \varnothing$.

(4) If x and y are two distinct points of X providing that $x \notin mCl^*_H(\{y\})$, then $mCl^*_H(\{y\}) \cap mCl(\{x\}) =\varnothing$.

(5) For each non empty subset A of X and each m^*_H-open set U with $A \cap U \neq \varnothing$, there exists an $m-$closed set F such that $A \cap F \neq \varnothing$ and $F \subseteq U$.

(6) Every m^*_H-open set U can be written as $U = \cup\{F : F \text{ is } m-closed \text{ and } F \subseteq U\}$.

(7) Every m^*_H-closed set F can be written as $F = \cap\{U : U \text{ is } m-open \text{ and } F \subseteq U\}$.

Proof. (1) $\Rightarrow$ (2) : Assume that F is an m^*_H-closed and $x \notin F$. Then $x \in X\backslash F$ and so by (1), we take $mCl(\{x\}) \subseteq X\backslash F$. It follows that $F \subseteq X\backslash mCl(\{x\}) = U$ (put). Then U is an $m-$open (here, we used that m has property ß) set in X, where $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3) : Let F be an m^*_H-closed set and $x \notin F$. Then by (2), there exists an $m-$open set U in X providing that $x \notin U$ and $F \subseteq U$. That is $U \cap (\{x\}) = \varnothing$ and $U \cap mCl(\{x\}) = \varnothing$ and thus $F \cap mCl(\{x\}) = \varnothing$.

(3) $\Rightarrow$ (4) : Let x and y be two distinct points of X with $x \notin mCl^*_H(\{y\})$. Since $mCl^*_H(\{y\})$ is an m^*_H-closed set and $x \notin mCl^*_H(\{y\})$, by (3), we get $mCl^*_H(\{y\}) \cap mCl(\{x\}) = \varnothing$.

(4) $\Rightarrow$ (1) : Let $x \in U$ and U be m^*_H-open. Attention that for every $y \notin U$, we get $x \notin mCl^*_H(\{y\})$ and thus by (4), $mCl(\{x\}) \cap mCl^*_H(\{y\}) = \varnothing$ for every $y \notin U$. It is seen that $mCl(\{x\}) \cap [\cup\{mCl^*_H(\{y\}) : y \in X\backslash U\}] = \varnothing$. Since U is m^*_H-open and $y \in X\backslash U$, then $\{y\} \subseteq mCl^*_H(\{y\}) \subseteq mCl^*_H(X\backslash U) = X\backslash U$. Therefore $X\backslash U = \cup\{mCl^*_H(\{y\}) : y \in X\backslash U\}$. Then $mCl(\{x\}) \cap (X\backslash U) = \varnothing$ and $mCl(\{x\}) \subseteq U$. So the hereditary $m-$space (X, m, H) is m^*-R_0.

(1) $\Rightarrow$ (5) : Let U be m^*_H-open and A a nonempty set of X providing that $A \cap U \neq \varnothing$. Get $x \in A \cap U$. Since X is an m^*-R_0 space and $x \in U$, $mCl(\{x\}) \subseteq U$. Let $mCl(\{x\}) = F$. Then it follows that F is an $m-$closed set with $F \subseteq U$ and $A \cap F \neq \varnothing$.

(5) $\Rightarrow$ (6) : Let U be m^*_H-open and $x \in U$. Then by (5) there exists an $m-$closed set N containing x and $N \subseteq U$. That is $x \in N \in \{F : F \text{ } is \text{ } m-$closed and $F \subseteq U\}$. Thus $U \subseteq \cup\{F : F$ *is* $m-closed$ *and* $F \subseteq U\} \subseteq U$.

(6) $\Rightarrow$ (7) : Obvious.

(7) $\Rightarrow$ (1) : Let U be an m^*_H-open set and $x \in U$. We have to prove that $mCl(\{x\}) \subseteq U$. Assume that there exists $y \in mCl(\{x\})$ providing that $y \notin U$. Because U is m^*_H-open and $y \notin U$, $mCl^*_H(\{y\}) \cap U = \varnothing$. Since $mCl^*_H(\{y\})$ is an m^*_H-closed set, by (7), we have $mCl^*_H(\{y\}) = \cap\{V :$ V *is* $m-open$ *in* X *and* $mCl^*_H(\{y\}) \subseteq V\}$. Since $mCl^*_H(\{y\}) \cap U = \varnothing$ and $x \in U$, $x \notin \cap\{V : V$ *is* $m-open$ *in* X and $mCl^*_H(\{y\}) \subseteq V\}$ and there exists an $m-$open set V in X providing that $x \notin V$ and $mCl^*_H(\{y\}) \subseteq V$. Since V is an $m-$open set containing y with $x \notin V$, $y \notin mCl(\{x\})$. This is a contradiction.

We describe a separation axiom called $m^* - R_1$ which is stronger than $m^* - R_0$.

Definition 3.6. *A hereditary* $m-$*space* (X, m, H) *is said to be* $m^* - R_1$ *if for every* $x, y \in X$ *with* $mCl(\{x\}) \neq mCl^*_H(\{y\})$, *there exist two disjoint* m^*_H-*open sets* U *and* V *providing that* $mCl(\{x\}) \subseteq U$ *and* $mCl^*_H(\{y\}) \subseteq V$.

Proposition 3.7. *If a hereditary* $m-$*space* (X, m, H) *is* $m^* - R_1$, *then it is* $m^* - R_0$.

Proof. Let $x \in U$ and U be m^*_H-open. We have to prove that $mCl(\{x\}) \subseteq U$. Assume that $y \notin U$. Then $mCl^*_H(\{y\}) \subseteq X \backslash U$ and thus $mCl(\{x\}) \neq mCl^*_H(\{y\})$. Since (X, m, H) is $m^* - R_1$, there exist two disjoint m^*_H-open sets V_x and V_y providing that $mCl(\{x\}) \subseteq V_x$ and $mCl^*_H(\{y\}) \subseteq V_y$. Therefore $y \notin mCl(\{x\})$. So it is seen that $mCl(\{x\}) \subseteq U$.

Remark 3.8. *We obtain the following implications for a hereditary* $m-$*space* (X, m, H).

$$m^* - R_1 \text{ space} \Rightarrow m^* - R_0 \text{ space} \Rightarrow m - R_0 \text{ space.}$$

Theorem 3.9. *For a hereditary* $m-$*space* (X, m, H), *the following statements are equivalent:*
(1) A hereditary $m-$*space* (X, m, H) *is an* $m^* - R_1$ *space.*
(2) For each $x, y \in X$ *one of the following holds:*
(i) If $U \in m^*_H$, *then* $x \in U$ *if and only if* $y \in U$.
(ii) There exist two disjoint m^*_H-*open sets* U *and* V *such that* $x \in U$ *and* $y \in V$.
(3) For each $x, y \in X$ *with* $mCl(\{x\}) \neq mCl^*_H(\{y\})$, *there exist* m^*_H*-closed sets* F_1 *and* F_2 *providing that* $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ *and* $X = F_1 \cup F_2$.

Proof. (1) $\Rightarrow$ (2) : For any pair of points $x, y \in X$, either $mCl(\{x\}) = mCl^*_H(\{y\})$ or $mCl(\{x\}) \neq mCl^*_H(\{y\})$,
(i) Let $mCl(\{x\}) = mCl^*_H(\{y\})$. If U is an m^*_H-open set containing x, then $y \in mCl(\{x\}) \subseteq U$ (since (X, m, H) is $m^* - R_0$). If U contains y, then $x \in mCl(\{x\}) \subseteq mCl(\{y\}) \subseteq U$.
(ii) Let $mCl(\{x\} \neq mCl^*_H(\{y\})$. By (1), there exist disjoint m^*_H-open sets U and V such that $x \in mCl(\{x\}) \subseteq U$ and $y \in mCl^*_H(\{y\}) \subseteq V$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ and $mCl(\{x\}) \neq mCl^*_H(\{y\})$. From here either $x \notin mCl^*_H(\{y\})$ or $y \notin mCl(\{x\})$. If $y \notin mCl(\{x\})$, then we get $y \notin mCl^*_H(\{x\})$. We pay attention to $x \notin mCl^*_H(\{y\})$. Then there exists an m^*_H-open set containing x but not containing y. Therefore by (2) , we have disjoint m^*_H-open sets U and V such that $x \in U$ and $y \in V$. Put $F_1 = X \backslash V$ and $F_2 = X \backslash U$. At that case F_1 and F_2 are m^*_H-closed sets providing that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

(3) $\Rightarrow$ (1) : Let x, y be two points of X providing that $mCl(\{x\}) \neq mCl^*_H(\{y\})$. Then using (3), there exist m^*_H-closed sets F_1 and F_2 providing that $x \in F_1$, $y \notin F_1$, $y \in F_2, x \notin F_2$ and $X = F_1 \cup F_2$. At that case $x \in F_1 \backslash F_2$ and $y \in F_2 \backslash F_1$, where $F_1 \backslash F_2$ and $F_2 \backslash F_1$ are disjoint m^*_H-open sets. Thus $mCl(\{x\}) \subseteq F_1 \backslash F_2$ and $mCl(\{y\}) \subseteq F_2 \backslash F_1$. Therefore it is seen that the hereditary $m-$space (X, m, H) is $m^* - R_1$ space.

Acknowledgements. The authors would like to thank the referee for his (her) helpful suggestions. The authors admit that the statements of some results in this paper were presented at the Third International Conference of Mathematical Sciences, Istanbul Turkey (ICMS 2019).

References

[1] Alimohammady, M. and Roohi, M. (2005). Fixed point in minimal spaces, Nonlinear Analysis Modelling and Cont. 10 (4): 305–314.

[2] Alimohammady, M. and Roohi, M. (2007). Linear minimal space, Chaos, Solitions & Fractals. 33(4): 1348–1354.

[3] Al-Omari, A. and Noiri, T. (2012). On $\psi*$-operator in ideal m-spaces, Bol. Soc. Paran. Mat. 30 (1): 53–66.

[4] Al-Omari, A. and Noiri, T. (2016). On operators in ideal minimal spaces, Mathematica. 58 (81): 3–13.

[5] Arivuoli, D. and Parimala, M. (2017). On generalised closed sets in ideal minimal spaces, Asia 2. Life Sciences, Suppl. 14 (1): 85–92.

[6] Buadong, S., Viriyapong, C. and Boonpok, C. (2011). On generalized topology and minimal structure spaces, Int. J. Math. Anal. 5 (31): 1507–1516.

[7] Cammaroto, F. and Noiri, T. (2005). On Λ_m-sets and related topological spaces, Acta Math. Hungar. 109(3): 261–279.

[8] Császár, A. (2007). Modification of generalized topologies via hereditary classes, Acta Math. Hungar. 115 (1-2): 29–35.

[9] Jafari, S., Rajesh, N. and Saranya, R. (2017). Semiopen sets in idealminimal spaces, "Vasile Alecsandri" Univer. Bacău Fac. Sci. Stud. Res. Ser. Math. Inform 27 (1): 33–48.

[10] Janković, D. and Hamlett, T. R. (1990). New topologies from old via ideals, Amer. Math. Month. 97: 295–310.

[11] Keun, W. K. and Kim, Y. K. (2011). MI-continuity and product minimal structure on minimal structure, Int. J. Pure Appl. Math. 69 (3): 329–339.

[12] Kuratowski, K. (1966). Topology, Vol. I, Academic Press, New York.

[13] Maki, H., Rao, K. C. and Gani, A. N. (1999). On generalized semi-open and preopen sets, Pure Appl. Math. Sci. 49: 17–29.

[14] Min, W. K. (2010). m-open sets and m-continuous functions, Commun. Korean Math. Soc. 25 (2): 251–256.

[15] Modak, S. (2013). Dense sets in weak structure and minimal structure, Commun. Korean Math. Soc., 28 (3): 589–596.

[16] Modak, S. (2018). Minimal spaces with a mathematical structure, J. Assoc. Arab Uni. Basic and App. Sci. 22: 98–101.

[17] Modak, S. (2013). Topology on grill m-space, Jordan J. Math. Stat. 6: 183–195.

[18] Modak, S. (2013). Operators on grill M-space, Bol. Soc. Paran. Mat. 31: 101–107.

[19] Noiri, T. (2007). A unified theory for modifications of g-closed sets, Rend. Circ. Mat. Palermo 56 (2): 171–184.

[20] Noiri, T. and Popa, V. (2002). On $m-D-$separation axioms. J. Math. Univ. Istanbul Fac. Sci. 61: 15–28.

[21] Noiri, T. and Popa, V. (2010). A decomposition of m-continuity, Bul. Univ. Petrol-Gaze Ploiesti, Mat.-Infor.-Fiz. 62(2): 46–53.

[22] Noiri, T. and Popa, V. (2018). Generalizations of closed sets in minimal spaces with hereditary classes, Annal. Univ. Sci. Budapest. 61: 69–83.

[23] Ozbakir, O. B. and Yildirim, E. D. (2009). On some closed sets in ideal minimal spaces, Acta Math. Hungar. 125 (3): 227–235.

[24] Parimala, M., Jeevitha, R. and Shanthakumari, S. (2017). On regular $MI-$ closed sets in minimal ideal topological spaces, J. Global Res. Math. Arch. 4 (10): 32–37.

[25] Parimala, M., Arivuoli, D., and Perumal, R. (2017). On some locally closed sets in ideal minimal spaces, Int. J. Pure Appl. Math. 113 (12): 230–238.

[26] Parimala, M. and Jeevitha, R. (2018). Minimal ideal $\alpha\psi$ submaximal in minimal structure spaces, Int. J. Math. Arch. 9 (11): 6–10.

[27] Parimala, M., Arivuoli, D. and Krithika, S. (2018). Separation axioms in ideal minimal spaces, Int. J. Rec. Tec. Eng. 7 (4S2).

[28] Popa, V. and Noiri, T. (2000). On M-continuous function. Anal. Univ. "Dunarea de Jos Galati", Ser. Mat. Fiz. Mec. Teor. Fasc. II. 18 (23): 31–41.

[29] Ravi, O., Tharmar, S. and Ganesan, S. (2011). Some locally closed sets in ideal minimal spaces, Int. J. Adv. Pure Appl. Math. 1 (1): 89–105.

[30] Vaśquez, L., Brown, M. S. and Rosas, E. (2011). Functions almost contra-super-continuity in m-spaces. Bol. Soc. Paran. Mat. 29 (2): 15–36.

[31] Loganathan, C., Vijaya Chandra, R. and Ravi, O. (2017). m^*- Operfect sets and $\alpha-m^*-$closed sets, Annal. Pure and Appl. Math. 13 (1): 131–141.

[32] Zakari, A. H. (2013). Some generalizations on generalized topology and minimal structure spaces. Int. J. Math. Anal. 7 (55): 2697–2708.

[33] Zakari, A. H. (2016). gm-continuity on generalized topology and minimal structure spaces, J. Assoc. Arab Univ. Basic Appl. Sci. 20: 78–83.