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The Type II-Topp-Leone-Gompertz-G Family of Distributions with Applications to COVID-19 Data

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Abstract. A new generalized family of distributions called the Type II-Topp-Leone-Gompertz-G (TII-TL-Gom-G) distribution is developed. We present some special cases of the introduced family of distributions. Various statistical properties of the new family of distributions were derived. Maximum likelihood estimates for the proposed family of models were obtained. A simulation study is carried out to evaluate the consistency of the maximum likelihood estimates. Two real data examples are used to demonstrate the usefulness of the new family of models.
Key Words and Phrases: Type II-Topp-Leone-G distribution, Gompertz Distribution, Distribution, Maximum Likelihood Estimation.

The Type II-Topp-Leone-G (TII-TL-G) family of distributions was developed by Elgarhy et al. [11]. One of its main objectives was to capture increasing, decreasing, J, and reverse-J shapes for the probability density and hazard rate functions. Moreover, the distribution has applications in the field of hydrology, as well as in analyzing a wide range of authentic and real-world situations. Some of the most useful generalizations of the Type II Topp-Leone distribution available in the literature include the Type II Topp-Leone Gumbel Type-2 distribution by Olayode et al. [21], Type II Topp-Leone generalized power Ishita distribution by Ikechukwu [15], Type II Topp-Leone inverse Weibull exponential distribution by Al-Marzouki [3] and Type II Topp-Leone Dagum by Sakthivel and Dhivakar [24]. All these families of distributions have been used for modeling real life data in various fields such as economics, engineering, biological studies, to name a few. The cumulative distribution function (cdf) of the TII-TL-G family of distributions is given by

$$F_{TII-TL-G}(x; b, \Psi) = 1 - [1 - G^2(x; \Psi)]^b \tag{1}$$

with shape parameter $b > 0$. The corresponding probability density function (pdf) is given by

$$f_{TII-TL-G}(x; b, \Psi) = 2bg(x; \Psi)G(x; \Psi)[1 - G^2(x; \Psi)]^{b-1}, \tag{2}$$

where $G(x, \Psi)$ is the baseline cdf with parameter vector Ψ. Gompertz distribution (Gompertz [13]) is a well-known distribution for modeling lifetime data in reliability, actuarial science and

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medical studies. Unfortunately, in practice often one needs to consider non-monotonic hazard rate function such as bathtub shape. This drawback has resulted in the development leading to the generalizations and extensions of some existing classical distributions. The Gompertz-G (Gom-G) family of distributions was proposed by Alizadeh et al. [2] and has cdf and pdf given by

$$F(x;\lambda,\gamma,\Psi) = 1 - \exp\left\{\frac{\lambda}{\gamma}\big[1-\big(1-G(x;\Psi)\big)^{-\gamma}\big]\right\}$$

and

$$f(x;\lambda,\gamma,\Psi) = \lambda g(x;\Psi)\big(1-G(x;\Psi)\big)^{-\gamma-1}\exp\left\{\frac{\lambda}{\gamma}\big[1-\big(1-G(x;\Psi)\big)^{-\gamma}\big]\right\},$$

respectively, for $\lambda,\gamma > 0$, where $G(x;\Psi)$ is the baseline cdf with parameter vector, and Ψ. In this note, we set $\lambda = 1$. Generalizations of the Gompertz distribution available in the literature includes the generalized Gompertz distribution by El-Gohary and Al-Otaibi [12], beta-Gompertz by Jafari et al. [17], Gompertz power series distribution by Jafari and Tahmasebi [16], Gompertz-G (Gom-G) family of distributions by Alizadeh et al. [20], a power Gompertz distribution by Ieren et al. [14] and Marshall-Olkin Gompertz-G family of distributions by Chipepa and Oluyede [6].

We introduce a new family of distributions, namely, the Type II-Topp-Leone-Gompertz-G (TII-TL-Gom-G) family of distributions. The motivations for the development for this new family of distributions is to generate distribution which is skewed, symmetric, J and reversed-J shaped, provide inevitably consistently good fit for real data. Moreso, the wide applications of the new family of distributions in different areas such as biology and actuarial sciences, is of tremendous practical importance.

In this note, the new family of distributions and its structural properties are introduced in Section 2. Maximum likelihood estimation is done in Section 3. Some of the special cases of the proposed family of distributions are discussed in Section 4. A simulation study is conducted in Section 5. Applications of the proposed model to real data examples are given in Section 6 and some concluding remarks in Section 7.

1. The Model and Structural Properties

In this work, we introduce the TII-TL-Gom-G family of distributions using the generalization proposed by Elgarhy et al. [11] and taking the baseline distribution to be the Gompertz-G family of distributions. Therefore, the cdf and pdf of the TII-TL-Gom-G family of distributions is given by

$$F_{TII-TL-Gom-G}(x;b,\gamma,\Psi) = 1 - \left(\left[1-\left(1-H(x;\Psi)\right)^2\right]^b\right) \tag{3}$$

and

$$\begin{aligned} f_{TII-TL-Gom-G}(x;b,\gamma,\Psi) &= 2bg(x;\Psi)\big(1-G(x;\Psi)\big)^{-\gamma-1}H(x;\Psi) \\ &\times \left(1-H(x;\Psi)\right)\left[1-\left(1-H(x;\Psi)\right)^2\right]^{b-1}, \end{aligned} \tag{4}$$

respectively, where $b,\gamma > 0$, Ψ is a vector of parameters from the baseline distribution cdf G and $H(x;\Psi) = \exp\left\{\frac{1}{\gamma}\big[1-\big(1-G(x;\Psi)\big)^{-\gamma}\big]\right\}$. The hazard rate function (hrf) is given by

$$\begin{aligned} h_{TII-TL-Gom-G}(x;b,\gamma,\Psi) &= 2bg(x;\Psi)\big(1-G(x;\Psi)\big)^{-\gamma-1}H(x;\Psi) \\ &\times \left(1-H(x;\Psi)\right)\left[1-\left(1-H(x;\Psi)\right)^2\right]^{-1}, \end{aligned}$$

for $b, \gamma, > 0$, and parameter vector Ψ.

1.1. Expansion of Density

We provide a series representation of the pdf of the TII-TL-Gom-G family of distributions in this subsection. Equation (4) can be written as

$$\begin{aligned} f_{TII-TL-Gom-G}(x;b,\gamma,\Psi) &= \sum_{i,j,k,l,w=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{i+j+l+w}(j+1)^k 2b}{\gamma^k k!(w+1)}\binom{b-1}{i}\binom{2i+1}{j} \\ &\times \binom{k}{l}\binom{-\gamma(l+1)-1}{w}(w+1)g(x;\Psi)G^w(x;\Psi) \\ &= \sum_{w=0}^{\infty}U_w g_w(x;\Psi), \end{aligned}$$

where

$$U_w = \sum_{i,j,k,l=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{i+j+l+w}(j+1)^k 2b}{\gamma^k k!(w+1)}\binom{b-1}{i}\binom{2i+1}{j}\binom{k}{l}\binom{-\gamma(l+1)-1}{w}, \tag{5}$$

and $g_w(x;\Psi) = (w+1)g(x;\Psi)G^w(x;\Psi)$ is an Exponentiated-G (Exp-G) distribution with power parameter $(w+1)$. Thus, the pdf of the TII-TL-Gom-G family of distributions can be expressed as an infinite linear combination of Exp-G distribution. See the appendix for derivations.

1.2. Stochastic Ordering

The concept of stochastic ordering is regularly used to display the ordering mechanism in probability models. For more details on stochastic ordering see Shaked and Shanthikumar [26]. A random variable X_1 is said to be stochastically smaller than X_2, $(X_1 <_{st} X_2)$ in the

- Stochastic order $(X_1 <_{st} X_2)$ if $F_{X_1}(x) \geq F_{X_2}(x)$ for all x.
- Hazard rate order $(X_1 <_{hr} X_2)$ if $h_{X_1}(x) \geq h_{X_2}(x)$ for all x.
- Likelihood ratio order $(X_1 <_{lr} X_2)$ if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreasing in x.

The stochastic orders given above are associated to each other and it holds that $X_1 <_{lr} X_2 \Rightarrow X_1 <_{hr} X_2 \Rightarrow X_1 <_{st} X_2$. Now, let $X_1 \sim TII-TL-Gom-G(b_1,\gamma,\Psi)$ and $X_2 \sim TII-TL-Gom-G(b_2,\gamma,\Psi)$. If $b_1 < b_2$, then the ratio $\frac{f_1(x;b_1,\gamma,\Psi)}{f_2(x;b_2,\gamma,\Psi)}$ is decreasing in x.

Then according to the definition of likelihood ratio ordering, is given by

$$\frac{f_1(x;b_1,\gamma,\Psi)}{f_2(x;b_2,\gamma,\Psi)} = \frac{b_1\left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left(1-G(x;\Psi)\right)^{-\gamma}\right]\right\}\right)^2\right]^{b_1-1}}{b_2\left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left(1-G(x;\Psi)\right)^{-\gamma}\right]\right\}\right)^2\right]^{b_2-1}}.$$

Taking derivative with respect to x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{f_1(x;b_1,\gamma,\Psi)}{f_2(x;b_2,\gamma,\Psi)}\right) = \frac{b_1}{b_2}(b_1-b_2)W^{b_1-b_2-1}W',$$

where $W = \left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left(1-G(x;\Psi)\right)^{-\gamma}\right]\right\}\right)^2\right]$ and $W' = \frac{dW}{dx}$.

If $b_1 < b_2$, then $\frac{\partial}{\partial x}\left(\frac{f_1(x;b_1,\gamma,\Psi)}{f_2(x;b_2,\gamma,\Psi)}\right) < 0$, which implies that $X_1 <_{lr} X_2$. Thus, X_1 and X_2 are stochastically ordered.

1.3. Quantile Function

The quantile function for the TII-TL-Gom-G family of distributions is given by

$$Q_X(u) = G^{-1}\left[1-\left(1-\gamma\ln\left[1-\left(1-\left[1-u\right]^{1/b}\right)^{1/2}\right]\right)^{-1/\gamma}\right].$$

Visit the appendix for the derivation.

1.4. Moments and Generating Function

If X follows the TII-TL-Gom-G distribution and $Y \sim$ Exp-G$(w+1)$, then the r^{th} moment, μ'_r of the TII-TL-Gom-G family of distributions is obtained as follows:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f_{TII-TL-Gom-G}(x;b,\gamma,\Psi)dx = \sum_{w=0}^{\infty} U_w E(Y^r),$$

where U_w is given by equation (5). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{w=0}^{\infty} U_w M_Y(t),$$

where $M_Y(t)$ is the mgf of Y and U_w is given by equation (5).

1.5. Mean Deviation, Bonferroni and Lorenz Curves

Deviation about the mean and deviation about the median are presented in this subsection. These measures are useful in the derivation of income inequality measures such as Bonferroni and

Lorenz curves, which are presented here as well. The deviation about the mean and about the median are defined as

$$\begin{aligned}\delta_1(x) &= \int_0^\infty |x-\mu| f_{TII-TL-Gom-G}(x;b,\gamma,\Psi)dx \\ &= 2\mu F_{TII-TL-Gom-G}(x;b,\gamma,\Psi)(\mu) - 2\mu + 2\sum_{w=0}^{\infty} U_w I_w^*(t)\end{aligned}$$

and

$$\delta_2(x) = \int_0^\infty |x-M| f_{TII-TL-Gom-G}(x;b,\gamma,\Psi)dx = -\mu + 2\sum_{w=0}^{\infty} U_w I_w^*(t),$$

respectively, where $\mu = E[X]$, M=Median(X), where $I_w^*(t) = \int_0^t x g_w(x;\Psi)dx$ is the first incomplete moment of the Exp-G distribution, and U_w is given in equation (5).

Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu}\sum_{w=0}^{\infty} U_w \int_0^q x g_w(x;\Psi)dx = \frac{1}{p\mu}\sum_{w=0}^{\infty} U_w I_w^*(t),$$

and

$$L(p) = \frac{1}{p}\sum_{w=0}^{\infty} U_w \int_0^q x g_w(x;\Psi)dx = \frac{1}{p}\sum_{w=0}^{\infty} U_w I_w^*(t),$$

respectively. 3D plots of skewness and kurtosis for the Type II-Topp-Leone-Gompertz-Burr XII (TII-TL-Gom-BXII) and Type II-Topp-Leone-Gompertz-Weibull (TII-TL-Gom-W) distributions are given in Figures 1 and 2. We observe that the TII-TL-Gom-BXII and TII-TL-Gom-W distributions can handle various levels of skewness and kurtosis when we fix the values of the different parameters.

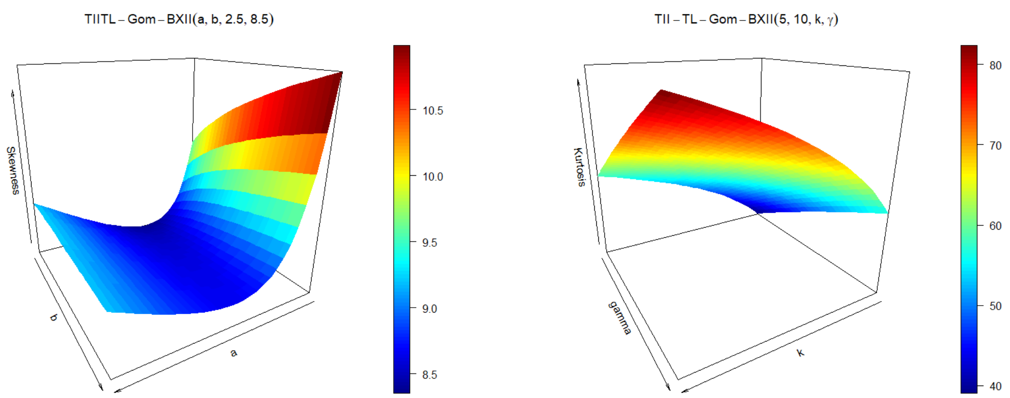


Figure 1: Plots of skewness and kurtosis for the TII-TL-Gom-BXII distribution

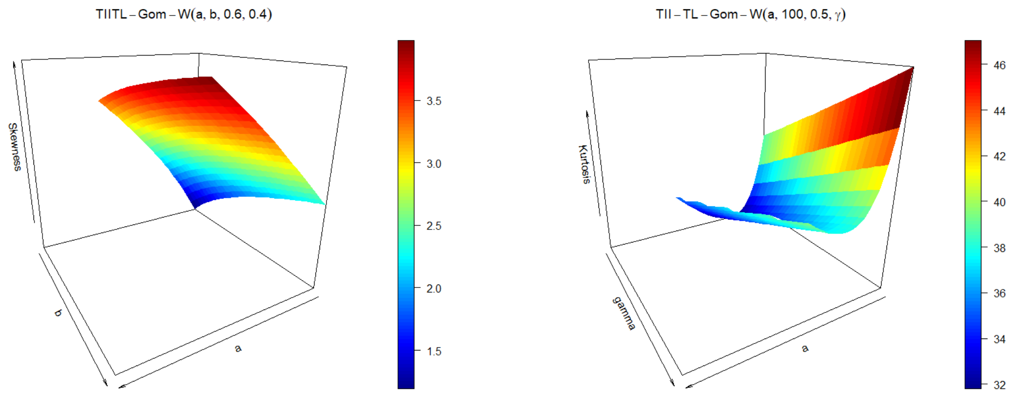


Figure 2: Plots of skewness and kurtosis for the TII-TL-Gom-W distribution

1.6. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the TII-TL-Gom-G family of distributions and suppose that $X_{1:n} < X_{2:n} < ... < X_{n:n}$ represents the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$\begin{aligned} f_{k:n}(x) &= \frac{n!}{(k-1)!(n-k)!}\sum_{i,j,p,q,r,s=0}^{\infty}\sum_{l=0}^{n-k}\frac{(-1)^{i+j+l+p+r+s}(p+1)^{q}2b}{\gamma^{q}q!(s+1)} \\ &\times \binom{n-k}{l}\binom{k+l-1}{i}\binom{b(i+1)-1}{j}\binom{2j+1}{p}\binom{q}{r}\binom{-\gamma(r+1)-1}{s} \\ &\times g(x;\Psi)(s+1)G^{s}(x;\Psi) \\ &= \sum_{s=0}^{\infty}\phi_s g_s(x;\Psi), \end{aligned} \tag{6}$$

where $g_s(x;\Psi) = (s+1)g(x;\Psi)G^s(x;\Psi)$ is an Exp-G distribution with power parameter $(s+1)$ and

$$\begin{aligned} \phi_s &= \frac{n!}{(k-1)!(n-k)!}\sum_{i,j,p,q,r=0}^{\infty}\sum_{l=0}^{n-k}\frac{(-1)^{i+j+l+p+r+s}(p+1)^{q}2b}{\gamma^{q}q!(s+1)} \\ &\times \binom{n-k}{l}\binom{k+l-1}{i}\binom{b(i+1)-1}{j}\binom{2j+1}{p}\binom{q}{r}\binom{-\gamma(r+1)-1}{s}. \end{aligned} \tag{7}$$

See the derivations in the appendix. The t^{th} moment of the distribution of the k^{th} order statistic from TII-TL-Gom-G family of distributions can be readily obtained from equation (7).

1.7. Probability Weighted Moments

The $(l,k)^{th}$ probability weighted moments (PWMs) of $X \sim$TII-TL-Gom-G is defined to be

$$\eta_{l,k} = E(X^l[F_X(X)]^k) = \int_{-\infty}^{\infty} x^l f_X(x)[F_X(x)]^k dx.$$

Using similar expansions from Section (1.6), we can write

$$\begin{aligned} f_X(x)[F_X(x)]^k &= \sum_{i,j,p,q,r,s=0}^{\infty} \frac{(-1)^{i+j+p+r+s}(p+1)^q 2b}{\gamma^q q!}\binom{k}{i}\binom{b(i+1)-1}{j} \\ &\times \binom{2j+1}{p}\binom{q}{r}\binom{-\gamma(r+1)-1}{s} g(x;\Psi)G^s(x;\Psi), \end{aligned}$$

which can be written as

$$f_X(x)[F_X(x)]^k = \sum_{s=0}^{\infty} b_s g_s(x;\Psi),$$

where

$$\begin{aligned} b_s &= \sum_{i,j,p,q,r=0}^{\infty} \frac{(-1)^{i+j+p+r+s}(p+1)^q 2b}{\gamma^q q!}\binom{k}{i}\binom{b(i+1)-1}{j} \\ &\times \binom{2j+1}{p}\binom{q}{r}\binom{-\gamma(r+1)-1}{s}. \end{aligned}$$

Consequently, the PWM of X reduces to

$$\eta_{l,k} = \sum_{s=0}^{\infty} b_s \int_{-\infty}^{\infty} x^l g_s(x;\Psi)dx = \sum_{s=0}^{\infty} b_s E(Y_s^l),$$

where $E(Y_s^l)$ is the l^{th} power of an Exp-G distribution with power parameter $(s+1)$.

1.8. Rényi Entropy

Rényi entropy [23] is an extension of Shannon entropy [25]. Rényi entropy for the TII-TL-Gom-G family of distributions is given by

$$I_R(\upsilon) = \frac{1}{1-\upsilon}\log\left(\sum_{p=0}^{\infty} \phi_p e^{(1-\upsilon)I_{REG}}\right),$$

$\upsilon > 0$, $\upsilon \neq 1$ where

$$I_{REG} = \frac{1}{1-\upsilon}\log\left(\int_0^{\infty}\left(\left[1+\frac{p}{\upsilon}\right] g(x;\Psi)G^{\frac{p}{\upsilon}}(x;\Psi)\right)^{\upsilon} dx\right),$$

is the Rényi entropy of the Exp-G distribution with power parameter $\left(1+\frac{p}{\upsilon}\right)$ and

$$\begin{aligned} \phi_p &= \sum_{i,j,k,l,z=0}^{\infty} \frac{(-1)^{i+j+l+p}(2b)^{\upsilon}(j+\upsilon)^k}{\gamma^k k!}\binom{\upsilon(b-1)}{i} \\ &\times \binom{2i+\upsilon}{j}\binom{k}{l}\binom{-\gamma(\upsilon+l)-\upsilon}{p}. \end{aligned}$$

Consequently, Rényi entropy of the TII-TL-Gom-G family of distributions can be readily derived from Rényi entropy of the Exp-G distribution. Visit the appendix to access the derivations.

2. Estimation

In this section, we apply the method of maximum likelihood estimation to estimate the parameter of the TII-TL-Gom-G family of distributions. Let $X_i \sim$ TII-TL-Gom-G(γ, b, Ψ) and $\mathbf{\Delta} = (\gamma, b, \Psi)^T$ be the vector of model parameters. The log-likelihood $\ell = \ell(\mathbf{\Delta})$ from a random sample of size n is

$$\begin{aligned}
\ell(\mathbf{\Delta}) &= n\log(2b) + \sum_{i=1}^{n}\log[g(x_i;\Psi)] - (\gamma+1)\sum_{i=1}^{n}\log[1-G(x_i;\Psi)] \\
&+ \sum_{i=1}^{n}\left(\frac{1}{\gamma}\left[1-\left(1-G(x_i;\Psi)\right)^{-\gamma}\right]\right) \\
&+ \sum_{i=1}^{n}\log\left[1-\exp\left\{\frac{1}{\gamma}\left[1-\left(1-G(x_i;\Psi)\right)^{-\gamma}\right]\right\}\right] \\
&+ (b-1)\sum_{i=1}^{n}\log\left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left(1-G(x_i;\Psi)\right)^{-\gamma}\right]\right\}\right)^2\right].
\end{aligned}$$

Elements of the score vector $U = (\frac{\partial\ell}{\partial\gamma}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\Psi_k})$ are given in the appendix. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\gamma}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\Psi_k})^T = \mathbf{0}$, using numerical methods such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+2}(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,\underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3. Some Special Cases of the TII-TL-Gom-G Family of Distributions

We consider some special cases of the TII-TL-Gom-G family of distributions, particularly when the baseline distributions are Burr XII, Kumaraswamy, Burr III and Weibull distributions, respectively. The cdf and pdf of the Burr XII distribution are given by $G(x;a,k) = 1-(1+x^a)^{-k}$ and $g(x;a,k) = akx^{a-1}(1+x^a)^{-k-1}$, respectively, for $a,k>0$ and $x>0$, for the Kumaraswamy distribution are given by $G(x;a,\theta) = 1-(1-x^a)^{\theta}$and $g(x;a,\theta) = a\theta x^{a-1}(1-x^a)^{\theta-1}$, respectively, for $a,\theta>0$ and $x>0$, for the Burr III distribution are given by $G(x;\alpha,\beta) = (1+x^{-\beta})^{-\alpha}$ and $g(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}$, respectively, for $\alpha,\beta>0$ and $x>0$, and for the Weibull distribution are given by $G(x;a) = 1-e^{-x^a}$ and $g(x;a) = ax^{a-1}e^{-x^a}$, respectively, for $a>0$ and $x>0$.

3.1. Type II-Topp-Leone-Gompertz-Burr XII (TII-TL-Gom-BXII) Distribution

The cdf and pdf of the TII-TL-Gom-BXII distribution are given by

$$F_{TII-TL-Gom-BXII}(x;a,b,k,\gamma) = 1-\left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-(1+x^a)^{k\gamma}\right]\right\}\right)^2\right]^b$$

and

$$f_{TII-TL-Gom-BXII}(x;a,b,k,\gamma) = 2bkax^{a-1}(1+x^a)^{-k\gamma-1}\exp\left\{\frac{1}{\gamma}\left[1-(1+x^a)^{k\gamma}\right]\right\}$$

$$
\begin{aligned}
& \times \left(1-\exp\left\{\frac{1}{\gamma}[1-(1+x^a)^{k\gamma}]\right\}\right) \\
& \times \left[1-\left(1-\exp\left\{\frac{1}{\gamma}[1-(1+x^a)^{k\gamma}]\right\}\right)^2\right]^{b-1},
\end{aligned}
$$

respectively for $\gamma, b, a, k > 0$.

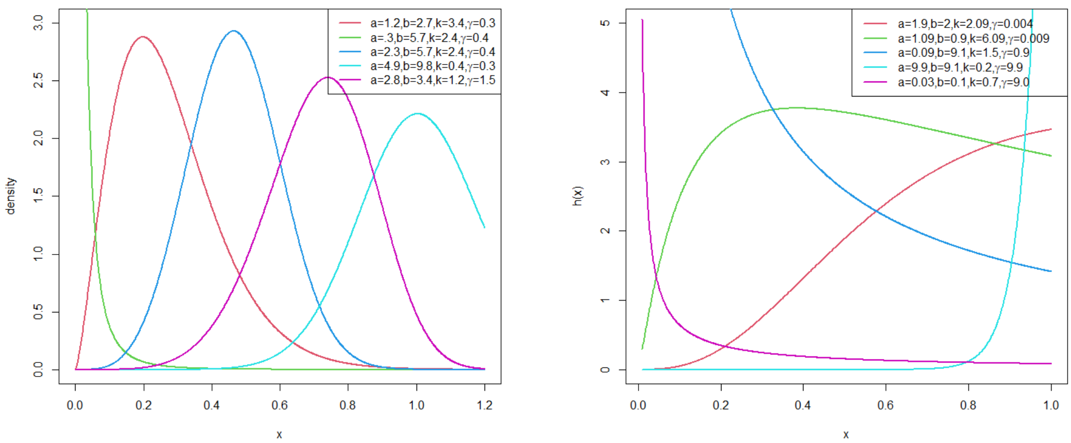


Figure 3: Plots of the pdf and hrf for the TII-TL-Gom-BXII distribution

Based on the Figure 3, we can observe that pdf of the TII-TL-Gom-BXII distribution can handle data that is almost symmetric, reverse-J, left or right-skewed. More so, the hrf of the distribution exhibit monotonic increasing, decreasing, J, reverse-J and upside down bathtub shapes. The Type II-Topp-Leone-Gompertz-Log-logistic (TII-TL-Gom-LLoG) and Type II-Topp-Leone-Gompertz-Lomax (TII-TL-Gom-Lomax) distributions are obtained when $k = 1$ and $a = 1$, respectively.

3.2. Type II-Topp-Leone-Gompertz-Kumaraswamy (TII-TL-Gom-K) Distribution

The cdf and pdf of the TII-TL-Gom-K distribution are given by

$$
F_{TII-TL-Gom-K}(x; a, b, \theta, \gamma) = 1-\left[1-\left(1-\exp\left\{\frac{1}{\gamma}[1-(1-x^a)^{-\theta\gamma}]\right\}\right)^2\right]^b
$$

and

$$
\begin{aligned}
f_{TII-TL-Gom-K}(x; a, b, \theta, \gamma) &= 2b\theta a x^{a-1}(1-x^a)^{-\theta\gamma-1}\exp\left\{\frac{1}{\gamma}[1-(1-x^a)^{-\theta\gamma}]\right\} \\
& \times \left(1-\exp\left\{\frac{1}{\gamma}[1-(1-x^a)^{-\theta\gamma}]\right\}\right) \\
& \times \left[1-\left(1-\exp\left\{\frac{1}{\gamma}[1-(1-x^a)^{-\theta\gamma}]\right\}\right)^2\right]^{b-1},
\end{aligned}
$$

respectively for $\gamma, b, a, \theta > 0$.

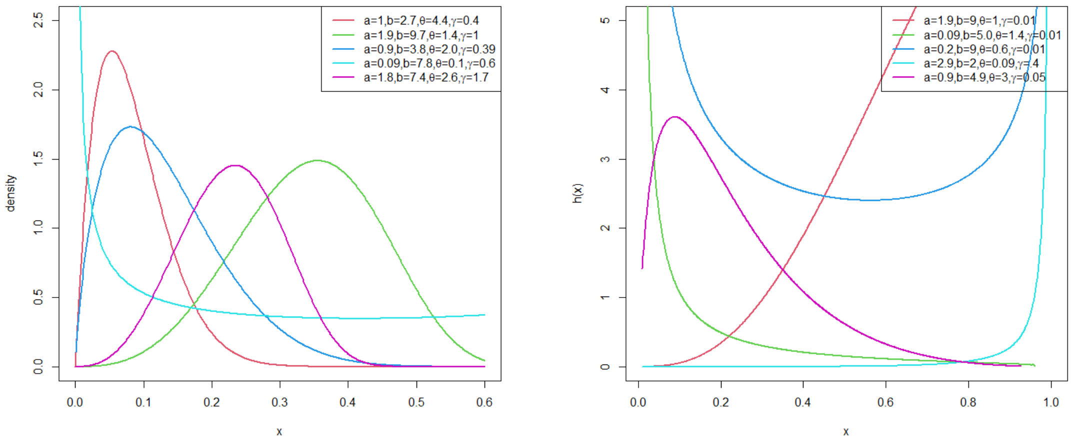


Figure 4: Plots of the pdf and hrf for the TII-TL-Gom-K distribution

The pdf of the TII-TL-Gom-K distribution can handle data that is almost symmetric, decreasing and heavy tailed. Also, the hrf of the distribution exhibit increasing, uni-modal, bathtub, upside bathtub and reverse-J shapes.

3.3. Type II-Topp-Leone-Gompertz-Burr III (TII-TL-Gom-BIII) Distribution

The cdf and pdf of the TII-TL-Gom-BIII distribution are given by

$$F_{TII-TL-Gom-BIII}(x;b,\alpha,\beta,\gamma) = \left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left[1-(1+x^{-\beta})^{-\alpha}\right]^{-\gamma}\right]\right\}\right)^2\right]^b$$

and

$$\begin{aligned} f_{TII-TL-Gom-BIII}(x;b,\alpha,\beta,\gamma) &= 2b\alpha\beta x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}\left[1-(1+x^{-\beta})^{-\alpha}\right]^{-\gamma-1} \\ &\times \exp\left\{\frac{1}{\gamma}\left[1-\left[1-(1+x^{-\beta})^{-\alpha}\right]^{-\gamma}\right]\right\} \\ &\times \left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left[1-(1+x^{-\beta})^{-\alpha}\right]^{-\gamma}\right]\right\}\right) \\ &\times \left[1-\left(1-\exp\left\{\frac{1}{\gamma}\left[1-\left[1-(1+x^{-\beta})^{-\alpha}\right]^{-\gamma}\right]\right\}\right)^2\right]^{b-1}, \end{aligned}$$

respectively for $b,\alpha,\beta,\gamma > 0$.

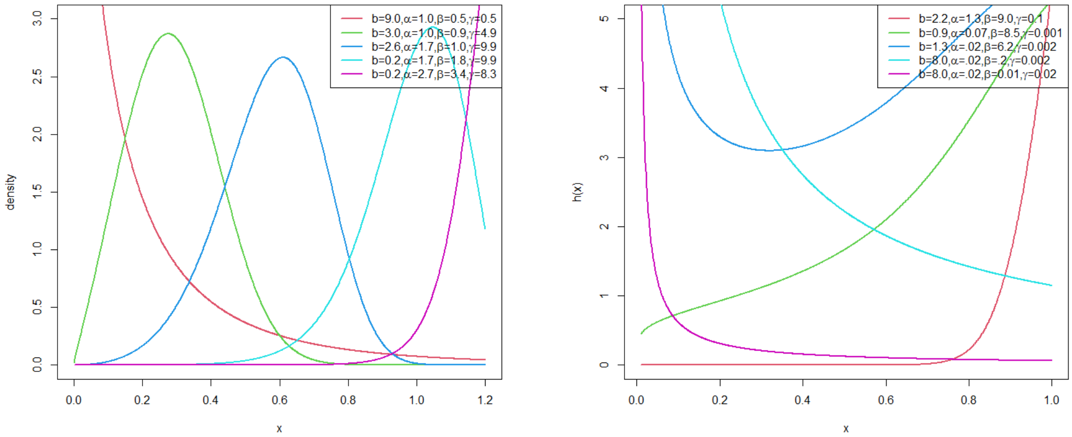


Figure 5: Plots of the pdf and hrf for the TII-TL-Gom-BIII distribution

The pdf of the TII-TL-Gom-BIII distribution can handle data that is almost symmetric, decreasing and heavy tailed. Also, the hrf of the distribution exhibit increasing, decreasing, bathtub, J and reverse-J shapes.

3.4. Type II-Topp-Leone-Gompertz-Weibull (TII-TL-Gom-W) Distribution

The cdf and pdf of the TII-TL-Gom-W distribution are given by

$$F_{TII-TL-Gom-W}(x;a,b,\gamma) = 1-\left[1-\left(1-\exp\left\{\frac{1}{\gamma}[1-e^{\gamma x^{a}}]\right\}\right)^{2}\right]^{b}$$

and

$$\begin{aligned} F_{TII-TL-Gom-W}(x;a,b,\gamma) &= 2abx^{a-1}e^{-x^{a}(\gamma-2)}\exp\left\{\frac{1}{\gamma}[1-e^{\gamma x^{a}}]\right\} \\ &\times \left(1-\exp\left\{\frac{1}{\gamma}[1-e^{\gamma x^{a}}]\right\}\right) \\ &\times \left[1-\left(1-\exp\left\{\frac{1}{\gamma}[1-e^{\gamma x^{a}}]\right\}\right)^{2}\right]^{b-1}, \end{aligned}$$

respectively for $a,b,\gamma>0$.

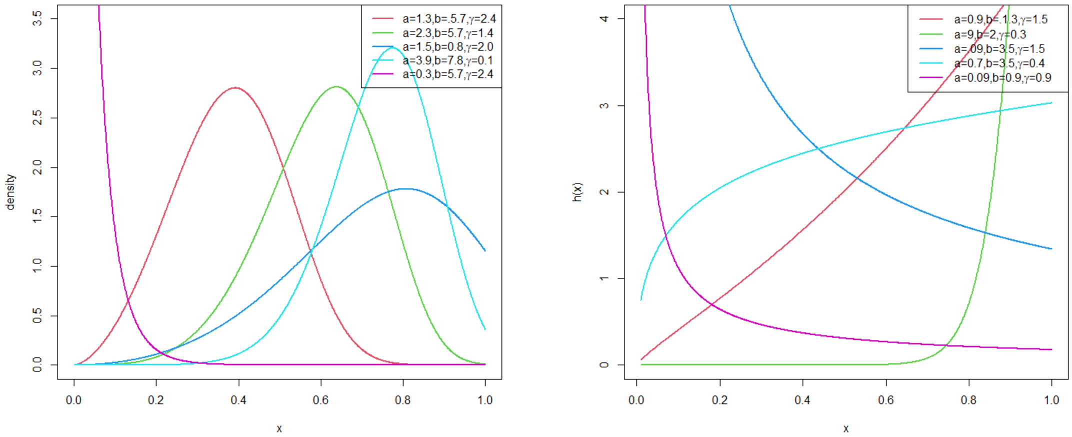


Figure 6: Plots of the pdf and hrf for the TII-TL-Gom-W distribution

Figure 6 shows that pdf of the TII-TL-Gom-W distribution can handle data that is symmetric, left or right-skewed. Also, the hrf of the distribution exhibit decreasing, increasing, J and reverse-J shapes.

4. Simulation Study

A simulation study to evaluate the consistency of the maximum likelihood estimates (MLEs) is carried out in this section. We simulated for N=1000 with sample sizes $n=$ 25, 50, 100, 200, 400, 800 and 1000. The performance of the MLEs was assessed performance using the mean, root mean square error (RMSE) and average bias. The average bias (ABias) and RMSE for the estimated parameter $\hat{\theta}$, are given by

$$ABias(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}},$$

respectively. We can conclude from Table 1 that our model gives consistent MLEs since the mean values approximate the true parameter values and the RMSE and bias decay for all the parameter values as the sample size increase.

Table 1: Monte Carlo Simulation Results for TII-TL-Gom-BXII Distribution: Mean, RMSE and Average Bias

	I	$a = 1.3$, $b = 0.5$, $k = 0.1$, $\gamma = 1.0$			II	$a = 0.9$, $b = 0.2$, $k = 0.2$, $\gamma = 0.8$	
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	2.577864	1.752896	1.277864	2.060045	1.736443	1.160045
	50	2.379014	1.684208	1.079014	1.573967	1.267806	0.673967
	100	2.220184	1.554578	0.920184	1.507548	1.320787	0.607548
a	200	2.063099	1.379965	0.763099	1.263494	0.903602	0.363494
	400	1.795539	1.131178	0.495539	0.964793	0.296305	0.064793
	800	1.496815	0.577145	0.196815	0.978491	0.334580	0.078491
	1000	1.419626	0.491880	0.119626	0.889787	0.065092	-0.010213
	25	1.378116	1.822252	0.878116	1.382401	2.241804	1.182401
	50	1.367883	1.740595	0.867883	1.051201	1.777073	0.851201
	100	1.239436	1.417708	0.739436	0.655399	1.023614	0.455399
b	200	1.157671	1.377538	0.657671	0.446557	0.648001	0.246557
	400	0.898201	0.891829	0.398201	0.261885	0.125869	0.061885
	800	0.739154	0.662808	0.239154	0.218838	0.054083	0.018838
	1000	0.661068	0.485370	0.161068	0.215228	0.057975	0.015228
	25	0.065392	0.104313	-0.034608	0.092592	0.156889	-0.107408
	50	0.075509	0.098674	-0.024491	0.123619	0.130617	-0.076381
	100	0.075860	0.083814	-0.024140	0.144282	0.124921	-0.055718
k	200	0.075744	0.066128	-0.024256	0.157426	0.089686	-0.042574
	400	0.086326	0.056008	-0.013674	0.178721	0.055139	-0.021279
	800	0.090790	0.040955	-0.009210	0.187602	0.040269	-0.012398
	1000	0.096096	0.038868	-0.003904	0.197215	0.018485	-0.002785
	25	1.423299	0.802515	0.423299	1.218103	0.714944	0.418103
	50	1.382848	0.680319	0.382848	1.159701	0.628261	0.359701
	100	1.353491	0.580524	0.353491	1.091537	0.530903	0.291537
γ	200	1.239778	0.440615	0.239778	0.952050	0.382512	0.152050
	400	1.128605	0.322495	0.128605	0.867844	0.153940	0.067844
	800	1.078886	0.242813	0.078886	0.814526	0.068486	0.014526
	1000	1.055474	0.207644	0.055474	0.808194	0.059059	0.008194
	III	$a = 0.4$, $b = 0.5$, $k = 0.4$, $\gamma = 1.6$			IV	$a = 1.5$, $b = 0.23$, $k = 0.3$, $\gamma = 0.9$	
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	1.372704	2.373685	0.972704	4.515457	4.833629	3.015457
	50	0.542608	0.531019	0.142608	3.320661	3.975213	1.820661
	100	0.449138	0.199440	0.049138	2.252255	2.017424	0.752255
a	200	0.418039	0.097148	0.018039	1.937119	1.430836	0.437119
	400	0.405900	0.063511	0.005900	1.599356	0.523352	0.099356
	800	0.402673	0.046527	0.002673	1.519662	0.220823	0.019662
	1000	0.400600	0.038290	0.000600	1.486214	0.185471	-0.013786
	25	1.642660	3.792367	1.142660	2.042800	2.885605	1.742800
	50	1.443009	2.165650	0.943009	1.484182	2.280288	1.184182
	100	1.454550	2.067744	0.954550	1.314657	2.272523	1.014657
b	200	1.200965	1.425104	0.700965	0.964425	1.363400	0.664425
	400	1.122699	1.236677	0.622699	0.685861	0.812307	0.385861
	800	0.943264	0.944601	0.443264	0.493777	0.484148	0.193777
	1000	0.907923	0.935429	0.407923	0.470948	0.444668	0.170948
	25	0.334445	0.335027	-0.065555	0.160941	0.239589	-0.139059
	50	0.347878	0.228021	-0.052122	0.207583	0.217178	-0.092417
	100	0.343267	0.210910	-0.056733	0.218241	0.169050	-0.081759
k	200	0.361599	0.189341	-0.038401	0.227354	0.150917	-0.072647
	400	0.363722	0.171079	-0.036279	0.243470	0.116865	-0.056530
	800	0.367394	0.149414	-0.032607	0.259640	0.084303	-0.040360
	1000	0.375977	0.141869	-0.024023	0.267250	0.077284	-0.032750
	25	1.969043	1.601236	0.369043	1.363633	0.850583	0.463632
	50	2.210601	1.654297	0.610601	1.374624	0.847430	0.474624
	100	2.122550	1.313174	0.522550	1.292143	0.697915	0.392143
γ	200	2.007864	1.019856	0.407864	1.200806	0.554280	0.300806
	400	1.958384	0.903729	0.358384	1.108577	0.429963	0.208577
	800	1.865204	0.706009	0.265204	1.032748	0.289528	0.132748
	1000	1.825939	0.658356	0.225939	1.016760	0.262228	0.116760

5. Applications

The usefulness of the TII-TL-Gom-BXII distribution is demonstrated in this section. To demonstrate the flexibility of the new family, we apply TII-TL-Gom-BXII distribution to two real data examples and compare it to several distributions. The perfomance of the model was examined using the goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-von Mises (W^*), Andersen-Darling (A^*) [5], Kolmogorov-Smirnov (K-S) and its p-value. The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is considered to be the best model.

We use R software via the nlm package to estimate model parameters. Tables 2 and 3 presents the estimates of the model parameters with their standard errors indicated in parenthesis, and the goodness-of-fit statistics for the two data sets. Also presented are the plots of the fitted densities histogram of the data and probability plots ([4]) to show how well our model fits the data sets.

We compare the TII-TL-Gom-BXII distribution to various models, namely, the exponential Lindley odd log-logistic Weibull (ELOLLW) by [18], beta-Weibull (BW) by [7], Kumaraswamy Weibull (KwW) by [8], odd exponentiated half-logistic Burr XII (OEHL-BXII) by [1], Topp-Leone-Marshall-Olkin-Weibull (TL-MO-W) by [9], Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by [10], Topp Leone-Gompertz-Weibull (TL-Gom-W) by [22] and Marshall Olkin-Gompertz-Weibull (MO-Gom-W) by [6]. The pdfs of the non-nested models are given in the appendix.

5.1. China Covid Data

This data set represents COVID-19 deaths in China for the period: 23 January 2020 to 28 March 2020 (further details can be found in https://www.worldometers.info/coronavirus/country/china). The data are: 8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 4, 6, 5, 3, 5.

Table 2: Parameter estimates and goodness-of-fit statistics for various models fitted for China covid data set

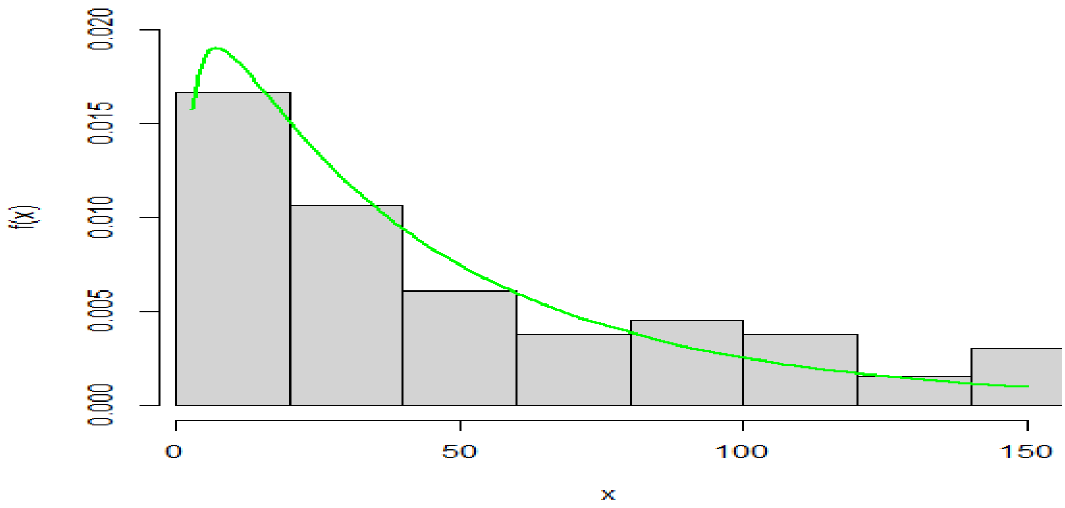
	Estimates				Statistics							
Model	a	b	k	γ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
TII-TL-Gom-BXII	5.6465	211.5400	1.2775×10^{-3}	58.7180	644.7	652.7	653.3	661.4	0.0818	0.6338	0.0877	0.6910
	(8.4400×10^{-9})	(5.2783×10^{-11})	(3.7384×10^{-5})	(4.4779×10^{-10})								
	γ	δ	λ	θ								
MO-Gom-W	0.0064	0.1869	1.1338	0.0024	643.4	651.4	652.1	660.2	0.0873	0.6493	0.0969	0.5648
	(0.0085)	(0.2062)	(0.2055)	(0.0034)								
	γ	b	λ									
TL-Gom-W	6.7498×10^{-10}	4.9095	0.1441	-	798.4	804.4	804.8	811.0	0.1209	0.8771	0.6175	2.2×10^{-16}
	(0.0056)	(0.6155)	(0.0121)									
	β	λ	θ	γ								
ELOLLW	0.5918	0.0087	2.8068	1.0746	646.9	654.9	655.6	663.7	0.1011	0.7611	0.0913	0.6414
	(1.5507)	(0.0035)	(0.3269)	(0.1251)								
	a	b	α	β								
BW	1.0204	2.8413	0.0075	1.0753	647.0	655.0	655.6	663.7	0.0998	0.7563	0.0916	0.6365
	(1.3478)	(0.0909)	(0.0045)	(0.9010)								
	a	b	α	β								
KwW	27.2730	381.7100	3.1087	0.0972	647.1	655.1	655.7	663.8	0.0931	0.7244	0.0918	0.6344
	(0.7874)	(3.7163×10^{-3})	(0.8705)	(7.9176×10^{-3})								
	α	λ	a	b								
OEHL-BXII	0.2786	2.4954×10^{-5}	3.5466	0.6525	661.6	669.6	670.2	678.3	0.1536	0.9967	0.1218	0.2815
	(0.0216)	(9.6561×10^{-6})	(4.0306×10^{-3})	(0.0236)								
	b	δ	λ	γ								
TL-MO-W	0.8102	1.1618	0.0038	1.2167	646.9	654.9	655.6	663.7	0.1082	0.7941	0.0894	0.6668
	(1.0422)	(1.0716)	(0.0204)	(1.0248)								
	a	b	λ	c								
KOL-LLoG	60.6250	1190.4000	2.2479	0.0593	647.0	655.0	655.6	663.7	0.0928	0.721	0.0927	0.6221
	(7.3754×10^{-4})	(2.8739×10^{-6})	(0.0469)	(5.4768×10^{-3})								

The estimated variance-covariance matrix for the TII-TL-Gom-BXII model for the China covid data is given by

$$\begin{bmatrix} 7.1234 \times 10^{-17} & 4.4549 \times 10^{19} & 3.1552 \times 10^{-13} & 3.7794 \times 10^{18} \\ 4.4549 \times 10^{19} & 2.7860 \times 10^{-21} & 1.9732 \times 10^{-15} & 2.3636 \times 10^{-20} \\ 3.1552 \times 10^{-13} & 1.9732 \times 10^{-15} & 1.3976 \times 10^{-9} & 1.6740 \times 10^{-14} \\ 3.7794 \times 10^{18} & 2.3636 \times 10^{-20} & 1.6740 \times 10^{-14} & 2.0052 \times 10^{-19} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$a \in [5.6465 \pm 1.6542 \times 10^{-8}]$, $b \in [211.5400 \pm 1.0345 \times 10^{-10}]$, $k \in [1.2775 \times 10^{-3} \pm 7.3273 \times 10^{-5}]$ and $\gamma \in [58.7180 \pm 8.7767 \times 10^{-10}]$.

From the results presented in Table 2, we conclude that the TII-TL-Gom-BXII model performs better than the other models considered on COVID-19 deaths in China since it has the lowest values for the goodness-of-fit statistics and K-S (and the largest p-value for the K-S statistic). The fitted density and probability plot in Figure 7 indicates how well the TII-TL-Gom-BXII model fits the data. Based on Figure 8, we can conclude that our model is performing well because the observed and fitted Kaplan-Meier curve, and the observed and empirical cumulative distribution function (ECDF) curve are close to each other respectively. Furthermore, based on the Total-Time-on-Test scaled plot, hazard rate function (HRF) plot in Figure 9 shows an upside down bathtub shape.



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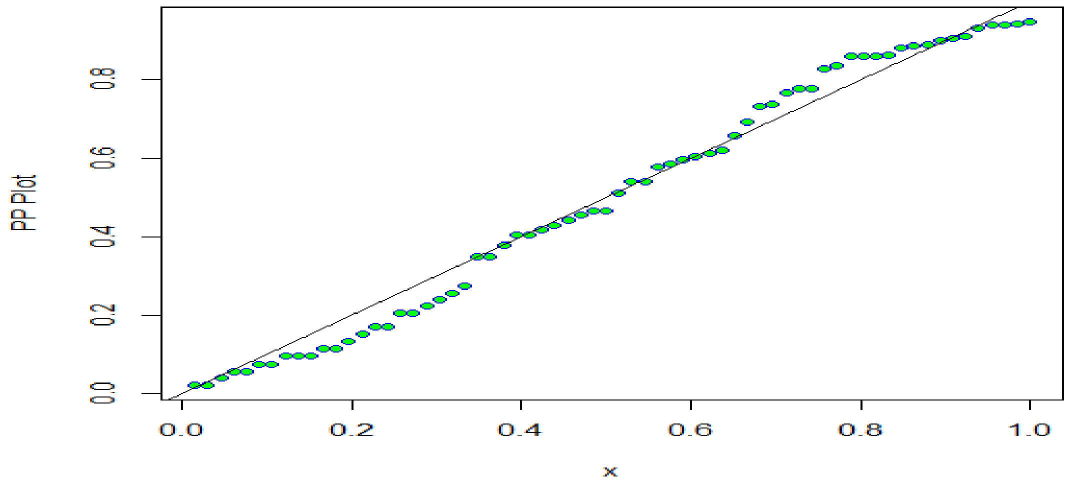

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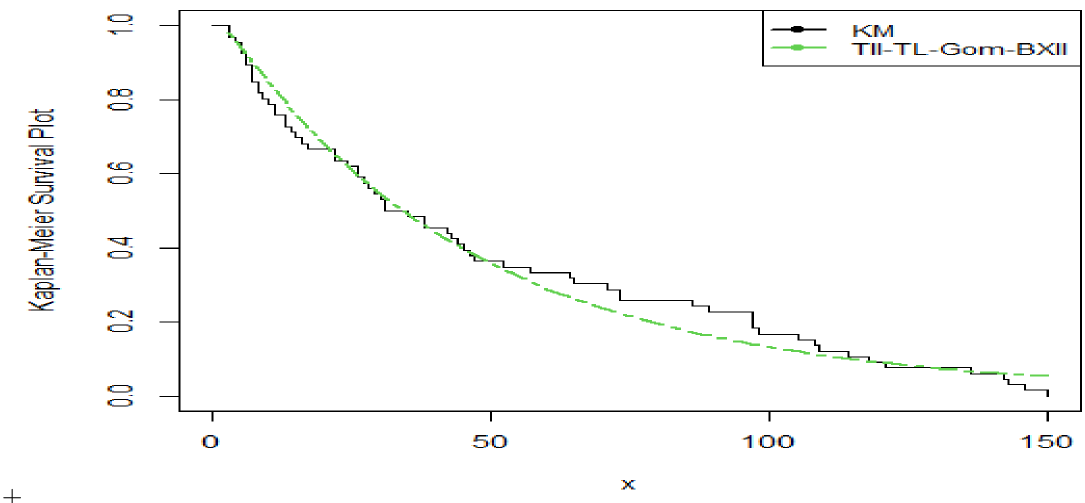
Figure 7: Fitted density and PP plots for China covid data



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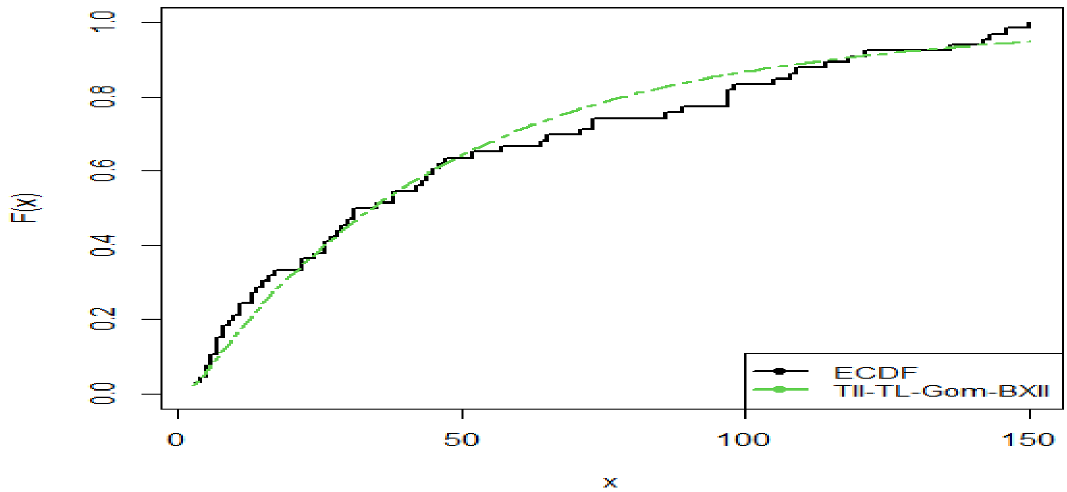

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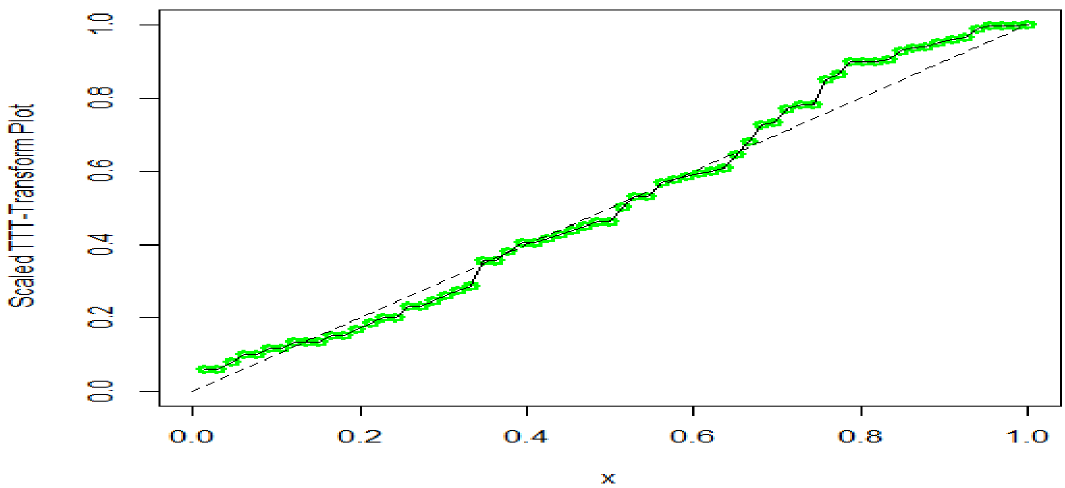
Figure 8: Fitted Kaplan-Meier and ECDF plots for China covid data



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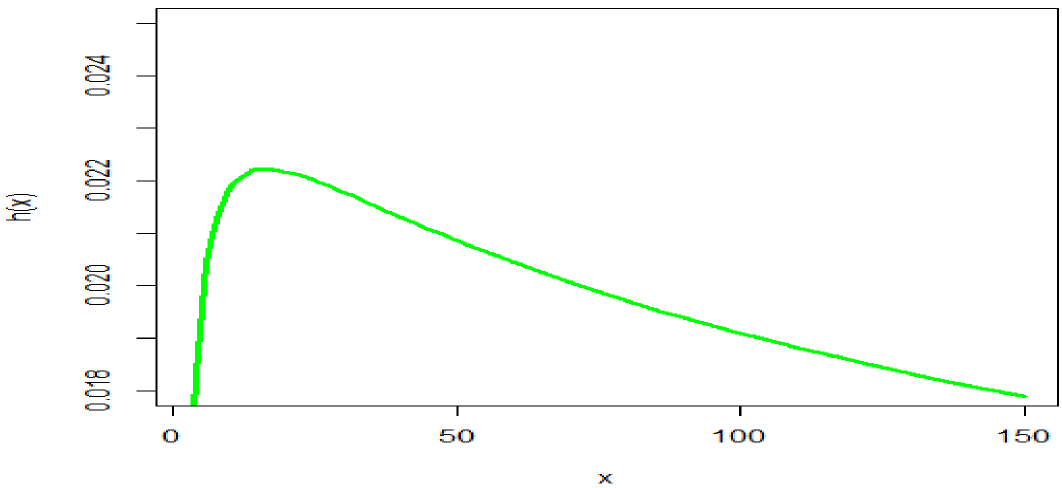

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Figure 9: Fitted TTT and HRF plots for China covid data

5.2. Europe Covid Data

We also considered the data set representing the number of daily deaths due to COVID-19 in Europe from the from 1st of March 2020 to 30th of March 2020, (see https://covid19.who.int/ for details). The observations are: 6, 18, 29, 28, 47, 55, 40, 150, 129, 184, 236, 237, 336, 219, 612, 434, 648, 706, 838, 1129, 1421, 118, 116, 1393, 1540, 1941, 2175, 2278, 2824, 2803, 2667.

Table 3: Parameter estimates and goodness-of-fit statistics for various models fitted for Europe covid data set

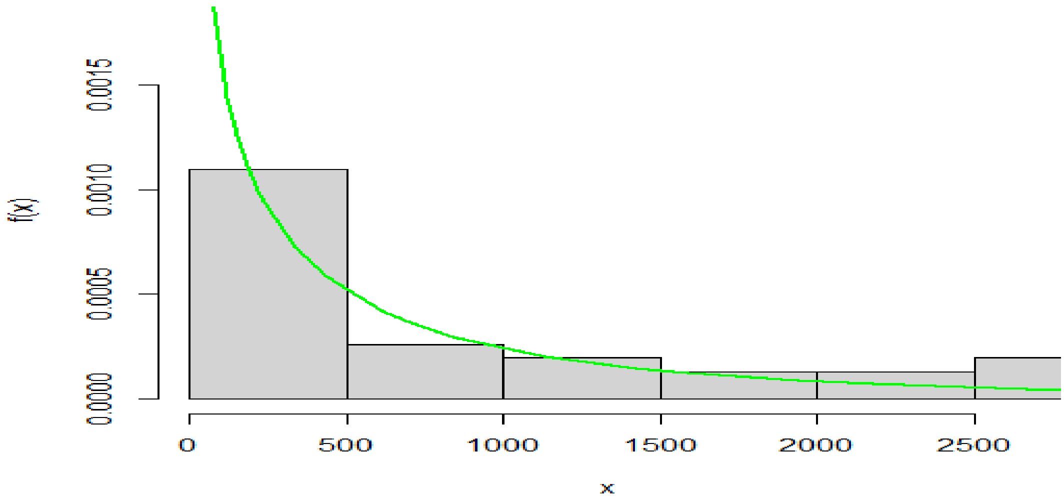
	Estimates				Statistics							
Model	a	b	k	γ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
TII-TL-Gom-BXII	6.9524	2067.1000	1.5986×10^{-4}	271.6800	471.9	479.9	481.4	485.6	0.0524	0.3797	0.1073	0.8311
	(1.3647×10^{-10})	(9.3613×10^{-14})	(5.9410×10^{-6})	(2.0884×10^{-12})								
	γ	δ	λ	θ								
MO-Gom-W	7.0213	0.4994	0.0755	4.4715×10^{-5}	472.6	480.6	482.2	486.4	0.0686	0.4679	0.1130	0.7821
	(5.3310×10^{-5})	(0.0001)	(0.0070)	(2.7604×10^{-5})								
	γ	b	λ									
TL-Gom-W	3.5190×10^{-9}	5.6821	0.0886	-	543.1	549.1	550.0	553.4	0.0661	0.4829	0.6079	1.616×10^{-11}
	(0.0056)	(0.6155)	(0.0121)									
	β	λ	θ	γ								
ELOLLW	0.5847	0.0006	2.1939	0.7190	472.5	480.5	482.0	486.2	0.0649	0.4462	0.1166	0.7498
	(1.5507)	(0.0035)	(0.3269)	(0.1251)								
	a	b	α	β								
BW	78.8250	46.3680	2.9394×10^{-3}	0.0711	474.5	482.5	484.1	488.3	0.0650	0.4754	0.1082	0.8230
	(3.6905×10^{-6})	(9.4243×10^{-7})	(8.7011×10^{-4})	(9.0839×10^{-3})								
	a	b	α	β								
KwW	33.8310	5325.2000	5.9308	0.0488	472.7	480.7	482.2	486.4	0.0602	0.4252	0.1079	0.8261
	(2.3280)	(8.5869×10^{-4})	(0.8534)	(5.2919×10^{-3})								
	α	λ	a	b								
OEHL-BXII	1.3189	975.0900	8.5154	2.9596×10^{-5}	542.3	550.3	551.8	556.0	0.0986	0.9967	0.3135	0.0033
	(3.7399×10^{-11})	(1.2863×10^{-13})	(1.4734×10^{-11})	(4.2416×10^{-6})								
	b	δ	λ	γ								
TL-MO-W	0.6683	1.7350	0.0016	0.8754	472.2	480.2	481.8	486.0	0.0734	0.4873	0.1370	0.5595
	(0.1931)	(0.0260)	(0.0019)	(0.1421)								
	a	b	λ	c								
KOL-LLoG	6.6911	1000.1000	0.4593	0.1095	472.7	480.7	482.2	486.4	0.0625	0.4363	0.1084	0.8216
	(3.2587×10^{-3})	(1.6286×10^{-6})	(0.0383)	(0.0142)								

The estimated variance-covariance matrix for TII-TL-Gom-BXII model on Europe covid data set is given by

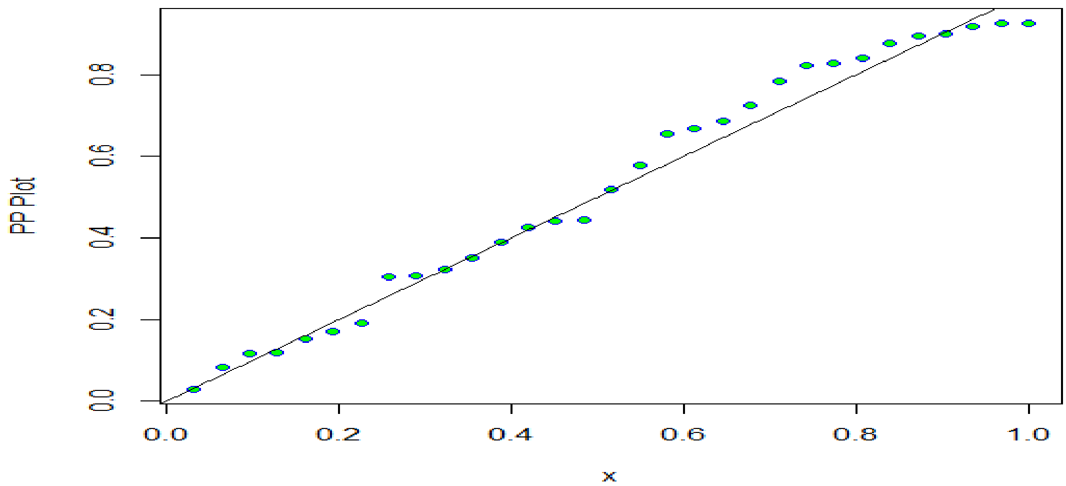
$$\begin{bmatrix} 1.8623\times10^{-20} & 1.2775\times10^{-23} & 8.1075\times10^{-16} & 2.8499\times10^{-22} \\ 1.2775\times10^{-23} & 8.7634\times10^{-27} & 5.5616\times10^{-19} & 1.9550\times10^{-25} \\ 8.1075\times10^{-16} & 5.5616\times10^{-19} & 3.5296\times10^{-11} & 1.2407\times10^{-17} \\ 2.8499\times10^{-22} & 1.9550\times10^{-25} & 1.2407\times10^{-17} & 4.3612\times10^{-24} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $a \in [6.9524 \pm 2.6747\times10^{-10}]$, $b \in [2067.1000 \pm 1.8348\times10^{-13}]$, $k \in [1.5986\times10^{-4} \pm 1.1644\times10^{-5}]$ and $\gamma \in [271.68 \pm 4.0932\times10^{-12}]$.

Based on the results shown in Table 3, we conclude that the TII-TL-Gom-BXII distribution performs better than the selected models considered on Europe covid data. In addition, Figure 10 displays the flexibility enjoyed on fitting the TII-TL-Gom-BXII distribution using the Europe covid data. Figures 11 and 12 indicate that our proposed model performs better since the observed fitted Kaplan-Meier and ECDF curves are each close to fitted model and the fitted hrf plot is exhibiting a decreasing shape.

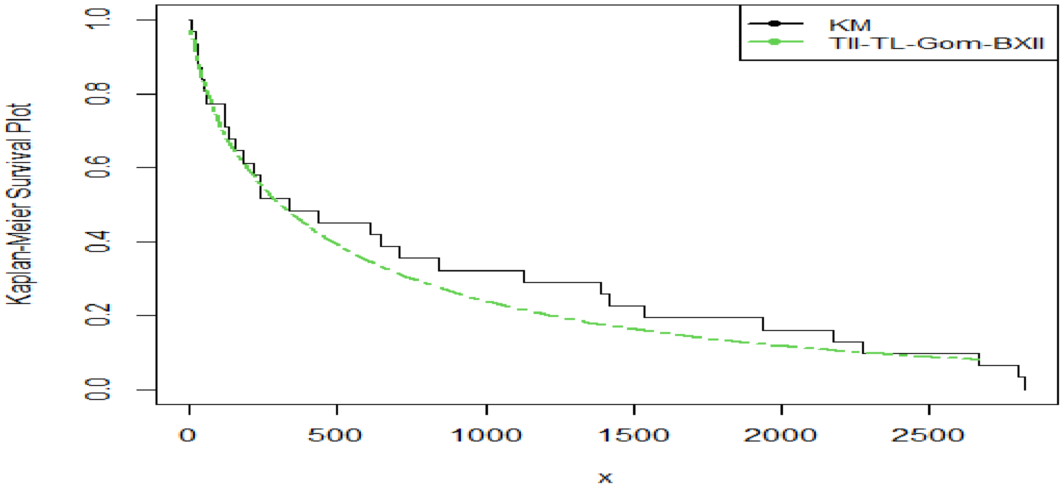


0.5

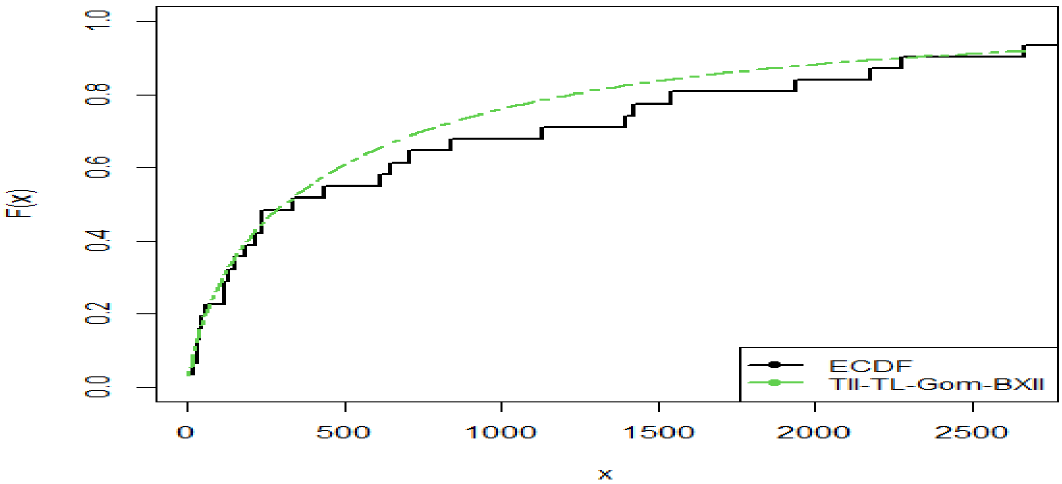


0.5

Figure 10: Fitted density and PP plots for Europe covid data

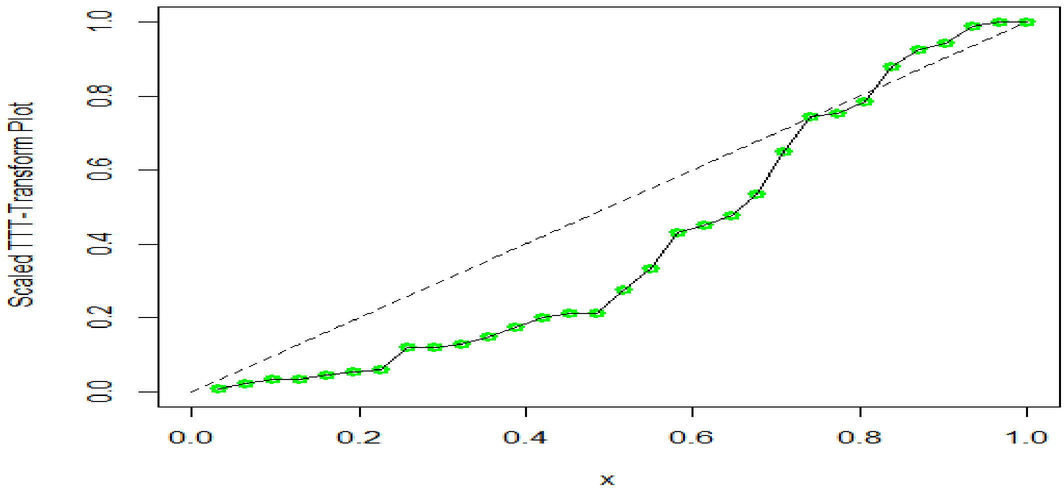


0.5

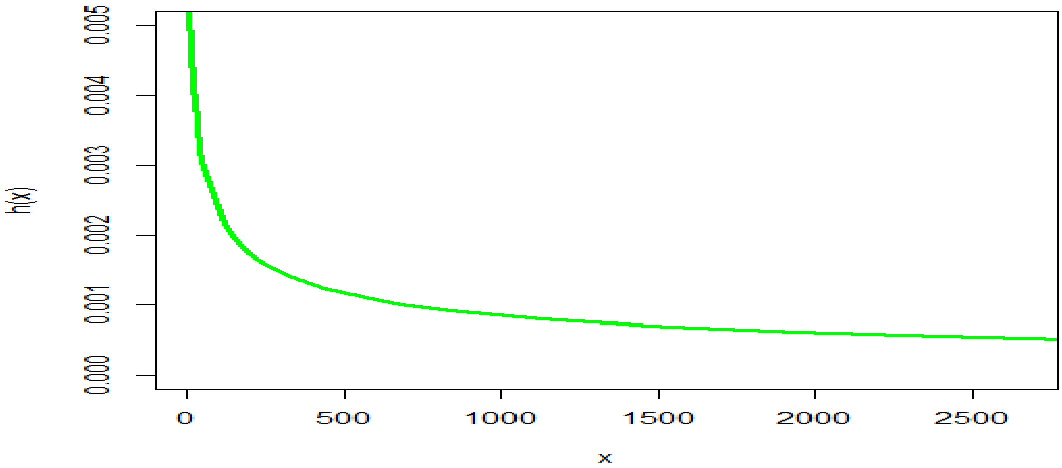


0.5

Figure 11: Fitted Kaplan-Meier and ECDF plots for Europe covid data



0.5



0.5

Figure 12: Fitted TTT and HRF plots for Europe covid data

6. Concluding Remarks

A new generalized family of distributions, called the Type II-Topp-Leone-Gompertz-G distribution was introduced. The hazard rate function of the proposed distribution can handle both monotonic and non-monotonic shapes. Statistical properties of the new distribution were also obtained. The method of maximum likelihood estimation was used to estimate the model parameters and a simulation study was conducted to examine the performance of the special case of the TII-TL-Gom-BXII distribution. The special case, the TII-TL-Gom-BXII distribution was applied to two real data sets to show the applicability and usefulness of the proposed family of distributions.

Appendix

The following URL contains derivations of statistical properties and elements of the score vector.
https://drive.google.com/file/d/1evTygI7hoNQUwSUML9z-$_{b}cz2jmwU4ZQ/view?usp = sharing$

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